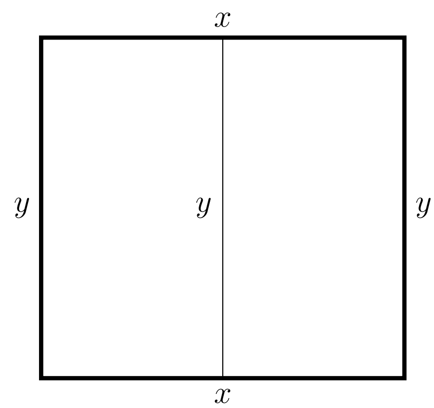
1. **(12 points)** *You are enclosing an entire rectangular field with chainlink fence, then placing a lightweight chicken-wire fence down the middle, parallel to the sides of the yard. The chainlink fence costs \$15 per foot to install, while the chickenwire will cost \$2 per foot. You have \$3840 for this project; what dimensions maximize the area you can enclose?*



In the above picture, we see the design drawn, with chainlink fences drawn with heavier lines. There are five fences, and we can see that their total cost in dollars will be $15x + 15x + 15y + 2y + 15y = 30x + 32y$. We are thus constrained both by the requirement that x and y be non-negative, and by the equation $30x + 32y = 3840$. Subject to these constraints we wish to maximize the area, which is clearly xy. Using the given constraint, we might note that $y = \frac{3840-30x}{32} = 120 - \frac{15}{16}x$, so the function we seek to maximize can be written as $A(x) = x\left(120 - \frac{15}{16}x\right) = 120x - \frac{15}{16}x^2$. The interval we're maximizing on will be $[0, 128]$, because if $x > 128$, then $y < 0$.

Note that $A'(x) = 120 - \frac{15}{8}x$, which has a critical point when $120 - \frac{15}{8}x = 0$; simple algebra will yield that $x = 64$. Thus we have three possible maximizing choices: $x = 0$, $x = 64$, and $x = 128$. $A(0) = A(128) = 0$, while $A(64) > 0$, so the optimum choice is to let $x = 64$ and $y = 120 - \frac{15}{16} \cdot 64 = 60$.

2. **(4 points)** *Using a starting point of $x_0 = 2$, show the first* three *steps of Newton's method to approximate a solution to $x^3 - 2x^2 + 4 = 0$; you do not need to arithmetically simplify your result for x_3.*

Recall that Newton's method is the formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. In this case, where our function is $x^3 - 2x^2 + 4$, the formula will be specifically $x_{n+1} = x_n - \frac{x_n^3-2x_n^2+4}{3x^2-4x}$. Running through three iterations:

$$
\begin{aligned}
x_0 &= 2 \\
x_1 &= 2 - \frac{2^3 - 2\cdot 2^2 + 4}{3\cdot 2^2 - 4\cdot 2} = 2 - \frac{4}{4} = 1 \\
x_2 &= 1 - \frac{1^3 - 2\cdot 1^2 + 4}{3\cdot 1^2 - 4\cdot 1} = 1 - \frac{3}{-1} = 4 \\
x_3 &= 4 - \frac{4^3 - 2\cdot 4^2 + 4}{3\cdot 4^2 - 4\cdot 4} = 4 - \frac{36}{32} = \frac{23}{8}
\end{aligned}
$$

Incidentally, this Newton's method procedure is surprisingly poor; it takes about 11 iterations before settling down at the zero of $x \approx -1.13$.

3. **(4 points)** *Using* $x_0 = 4$ *as your starting point, use* two *steps of Newton's method to approximate* $\sqrt[3]{100}$. *You do not need to (and probably do not want to) arithmetically simplify your result for* x_2.

   $\sqrt[3]{100}$ is a root of the polynomial $x^3 - 100$ (there are others, but this is the most obvious choice), so we want to use Newton's method specifically incarnated as the rule $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x^3-100}{3x^2}$. Running this twice:

$$\begin{aligned} x_0 &= 4 \\ x_1 &= 4 - \frac{4^3 - 100}{3 \cdot 4^2} = 4 - \frac{-36}{48} = \frac{19}{4} = 4.75 \\ x_2 &= 4 - \frac{\left(\frac{19}{4}\right)^3 - 100}{3\left(\frac{19}{4}\right)^2} \end{aligned}$$

   This last expression is $\frac{3353}{722}$ or approximately 4.644044, which you couldn't reasonably calculate. Note that the actual value of $\sqrt[3]{100}$ is approximately 4.641589, so this approximation is good but not great.

4. **(2 point bonus)** *Explain, using graphs and descriptions of the transformations performed on those graphs, why the linear approximation of* $\sqrt[k]{n}$ *from a reference point* a *gives the same result as the first step of Newton's method on an appropriate graph with* $x_0 = a$.