1. Let U be a nonempty set and let R be the set of all subsets of U (i.e. the power set of U). For the two given proposed definitions of "addition" and "multiplication", determine whether R is a ring or not; if it is not a ring, explain why, and if it is a ring, identify its identity elements.
   (a) $A + B = A \cup B$ and $A \cdot B = A \cap B$.
   (b) $A + B = (A \cup B) - (A \cap B)$ (also known as the "symmetric difference") and $A \cdot B = A \cap B$.
2. Prove that for a commutative ring R and $x \in R$, x is a unit of R if and only if x divides every element of R.
3. Prove the following two statements:
   (a) For any ring R and a (not necessarily finite) collection $\mathcal{R}$ of subrings of R, their intersection
   $$\bigcap_{S \in \mathcal{R}} S$$
   is also a subring of R. (Hint: you may find it easier to warm up by addressing just an intersection of two subrings of R)
   (b) For a ring R and a subset S of R, there is a unique *smallest* subring $\langle S \rangle$ of R which contains all the elements of S (hint: choose an appropriate collection $\mathcal{R}$ for the above statement).
4. Address the two following questions about subrings:
   (a) Let $\mathbb{Q}[\sqrt{2}, \sqrt{3}]$ denote the smallest subring of $\mathbb{R}$ which contains every rational number, the irrational number $\sqrt{2}$, and the irrational number $\sqrt{3}$. Let $\mathbb{Q}[\sqrt{2}+\sqrt{3}]$ denote the smallest subring of $\mathbb{R}$ which contains every rational number and the irrational number $\sqrt{2} + \sqrt{3}$ (note that the previous question guarantees that these structures are well-defined). Prove that $\mathbb{Q}[\sqrt{2}, \sqrt{3}] = \mathbb{Q}[\sqrt{2} + \sqrt{3}]$.
   (b) Let $\mathbb{Z}[\sqrt{2}, \sqrt{3}]$ and $\mathbb{Z}[\sqrt{2}+\sqrt{3}]$ be defined as in the previous part except with "rational number" replaced by "integer" in the definition. Demonstrate that $\mathbb{Z}[\sqrt{2}, \sqrt{3}] \neq \mathbb{Z}[\sqrt{2} + \sqrt{3}]$.
5. Prove that if the additive group of a ring R is cyclic, then R is commutative.

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