1. *Prove that, for an integral domain R, the ring of polynomials $R[x]$ is an integral domain, and that the units in $R[x]$ are exactly the units in R.*

   Proof. Suppose contrariwise that $R[x]$ has nonzero zero divisors; let $f(x), g(x) \in R[x]$ thus be nonzero polynomials of degree n and m respectively such that $f(x)g(x) = 0$.. Thus $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and $g(x) = b_m x^m + \cdots + b_0$, where each $a_i, b_j \in R$. Then the coefficient of x^{n+m} in $f(x)g(x)$ is $a_n b_m$; since $f(x)g(x) = 0$ as polynomials, each coefficient of $f(x)g(x)$ is zero, and specifically $a_n b_m = 0$. Since R is an integral domain, either $a_n = 0$ or $b_m = 0$, contradicting the degree criteria for $f(x)$ or $g(x)$.

   In addition, let us show that no nonconstant element of $R[x]$ can be a unit. Because R is an integral domain, with $f(x)$ and $g(x)$ defined as above, $f(x)g(x)$ will have a nonzero coefficient for x^{m+n}, so the degree of a product is the sum of the degrees of the factors. Since 1 is a polynomial of degree zero, $f(x)g(x) = 1$ only if $f(x)$ and $g(x)$ are themselves polynomials of degree zero; i.e., constant polynomials. So in order for them to be inverses, $f(x)$ and $g(x)$ must be equal to some a and b respectively with $a, b \in R$, and $ab = 1$; which is to say, $f(x)$ is a unit if and only if it is equal to a degree-zero polynomial whose constant term is a unit. □

2. *Prove that the ring $\mathbb{Z}_2[x]/\langle x^3 + x + 1\rangle$ is a field, but that $\mathbb{Z}_3[x]/\langle x^3 + x + 1\rangle$ is not a field.*

   Proof. The easiest way to prove this is to show that $\langle x^3 + x + 1\rangle$ is a maximal ideal in $\mathbb{Z}_2[x]$ but not in $\mathbb{Z}_3$, and appeal to the known result that R/I is a field iff I is maximal in R.

   For our former assertion, suppose $\langle x^3 + x + 1\rangle \subsetneq I \subsetneq \mathbb{Z}_2[x]$. Thus, there must be at least one element of I which is not in $\langle x^3 + x + 1\rangle$, but $1 \notin I$, since any ideal containing 1 would contain all of $\mathbb{Z}_2[x]$. We shall show that no such I can exist, and thus that $\langle x^3 + x + 1\rangle$ is in fact maximal. Let $f(x)$ be an element of I which is not a multiple of $\langle x^3 + x + 1\rangle$ and which is of the least degree possible under that condition. We shall see by process of elimination below that $f(x)$ could only be a linear polynomial, an that neither of the linear polynomials are in fact possible additions ot the ideal:

   Case I: $f(x)$ has degree 0. Since $f(x)$ is a nonzero constant, then $f(x) = 1$ and then $I = \mathbb{Z}_2[x]$, contradicting its propriety as an ideal.

   Case II: $f(x)$ has degree of 3 or more. Then $f(x)$ and $x^{\deg f - 3}(x^3 + x + 1)$ are polynomials with the same leading term, so $f(x) - x^{\deg f - 3}(x^3 + x + 1)$ will be an element of I of lower degree, and since $f(x)$ was not a multiple of $x^3 + x + 1$, neither is $x^{\deg f - 3}(x^3 + x + 1)$, contradicting our claim that $f(x)$ had the lowest degree possible.

   Case III: $f(x)$ has degree 2. $f(x)$ will have leading term x^2, so that $x^3 + x + 1 - xf(x)$ is an element of I of degree less than 3, and it will be nonzero, since its constant term is nonzero. If $x^3 + x + 1 - xf(x)$ has degree less than 2, that contradicts our assertion that $f(x)$ is the nonzero element of I of least degree. If the degree of $x^3 + x + 1 - xf(x)$ is exactly 2, on the other hand, then $x^3 + x + 1 - xf(x) - f(x) = x^3 + x + 1 - (x+1)f(x)$ will itself be an element of I of degree less than 2, since the leading terms of x^2 will cancel. Note that since $x^3 + x + 1 - (x+1)f(x)$ evaluated at $x = 1$ is nonzero, it is a nonzero polynomial, and since it is of degree less than 3, it is not a multiple of $x^3 + x + 1$; thus, as an element of I of degree 1, it contradicts our assertion that $f(x)$ has the lowest degree possible.

   Case IV: $f(x)$ has degree 1. There are exactly two possibilities: either $f(x) = x$ or $f(x) = x + 1$. If $f(x) = x$, then $x^3 + x + 1 - (x^2 + 1)f(x) = 1$, contradicting $f(x)$'s minimality. If $f(x) = x + 1$, then $x^3 + x + 1 - (x^2 + x)f(x) = 1$, again contradicting $f(x)$'s minimality.

   Thus we see that no ideal can lie strictly in inclusion between $\langle x^3 + x + 1\rangle$ and the whole ring in $\mathbb{Z}_2[x]$, so $\langle x^3 + x + 1\rangle$ is maximal.

For the latter assertion, note that since, in Z_3, $x^3+x+1=(x+2)(x^2+x+2)$ so $\langle x^3+x+1\rangle \subsetneq \langle x+2\rangle \subsetneq \mathbb{Z}_3[x]$, demonstrating non-maximality. □

3. *Given a ring homomorphism $\varphi : R \to R'$ on commutative rings, let $\varphi^{-1}(S) = \{r \in R : \varphi(r) \in S\}$. Note that φ^{-1} will map ideals to ideals.*

   (a) *Prove that if I' is a prime ideal of R', then $\varphi^{-1}(I')$ is a prime ideal of R.*

   Proof. For brevity, let us define $I = \varphi^{-1}(I')$, and prove the contrapositive: that if I is nonprime, then I' is nonprime as well. From nonprimeness of I, there exist $a, b \in R - I$ such that $ab \in I$. Since $a, b \notin I$, it follows from the definition of φ^{-1} that $\varphi(a)$ and $\varphi(b)$ are not in I'. Likewise, from the definition of φ^{-1}, since $ab \in \varphi^{-1}(I')$, $\varphi(ab) \in I'$. Since $\varphi(ab) = \varphi(a)\varphi(b)$, we can express this element of I' as a product of two demonstrated non-elements of I', so I' is not prime. □

   (b) *Demonstrate that if I' is a maximal ideal of R', then $\varphi^{-1}(I')$ need not be a maximal ideal of R.*

   One straightforward possibility is that $\varphi^{-1}(I')$ may not be a proper ideal of R at all; for instance, if $R = 7\mathbb{Z}$, $R' = \mathbb{Z}$, and φ is the inclusion map, then $I' = \langle 7\rangle$ is a maximal ideal whose pullback is the entirety of R, and thus not maximal because it is not proper.

4. *Prove that for ideals I and J of a commutative ring, and IJ representing the set consisting of every product of terms from I and J, it is the case that $IJ \subseteq I \cap J$. Under what circumstances will $IJ = I \cap J$?*

   Proof. For every $j \in J$, we know $Ij \subseteq I$ by the absorption property of ideals, so $IJ = \bigcup_{j\in J} Ij \subseteq I$. Likewise each $iJ \subseteq J$ so $IJ \subseteq J$. Then since IJ is a subset of both I and J, it is a subset of $I \cap J$. □

   Clearly, if either I or J is the entire ring or trivial, then $IJ = I \cap J$. In general, however, if neither I nor J is trivial or the entire ring, IJ will not necessarily equal $I \cap J$. Note that in $\mathbb{Z}$, for instance, $\langle a\rangle\langle b\rangle = \langle ab\rangle$, while in general $\langle a\rangle \cap \langle b\rangle = \langle \mathrm{lcm}(a, b)\rangle$, and these expressions are only the same if a and b are relatively prime.

5. *For any subset A of a commutative ring R, the annihilator $\mathrm{Ann}(A)$ of A is the set of all $r \in R$ such that $ra = 0$ for all $a \in A$. Prove that $\mathrm{Ann}(A)$ is always an ideal in R.*

   Proof. Let us show first $\mathrm{Ann}(A)$ is a subring of R. Note that $0a = 0$ is always true, so $0 \in \mathrm{Ann}(A)$. If $xa = 0$ and $ya = 0$ for every $a \in A$, it is also the case that $(x + y)a = xa + ya = 0 + 0 = 0$, $(-x)a = -(xa) = -0 = 0$, and $(xy)a = x(ya) = x0 = 0$, so $x + y$, $-x$, and xy are also in $\mathrm{Ann}(A)$, guaranteeing the closure and additive inverse properties.

   Now we need simply show that $\mathrm{Ann}(A)$ is absorbing, and that for any $r \in R$ and $x \in \mathrm{Ann}(A)$, $rx \in \mathrm{Ann}(A)$. This will be true because by definition $xa = 0$ for all $a \in A$, and so $(rx)a = r(xa) = r0 = 0$. □