1. For each of the following rings R and subsets S thereof, determine whether S is a subring of R.

   (a) $R = \mathbb{Q}$; S is the set of all rational numbers whose denominators are not divisible by 3.

   (b) R is the ring of all real 2×2 matrices; S is the set of all matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ for real numbers a and b.

2. Prove the two following facts about ideal intersections.

   (a) If I and J are ideals of a commutative ring R, then $I \cap J$ is also an ideal of R.

   (b) If I and J are both *prime* ideals of a ring R, then $I \cap J$ is also a *prime* ideal of R (you may, of course, use the result from the previous part in this question).

3. Let the homomorphism $\varphi : \mathbb{Z}[x] \to \mathbb{R}$ be defined by the mappings $\varphi(1) = 1$ and $\varphi(x) = 1 + \sqrt{2}$. Describe $\ker \varphi$.

4. For rings R and S, the direct product ring $R \oplus S$ consits of ordered pairs from $R \times S$, with termwise operations: $(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$ and $(r_1, s_1)(r_2, s_2) = (r_1 r_2, s_1 s_2)$. Prove that if R and S both have at least 2 elements, then $R \oplus S$ is not an integral domain.

5. Let φ be a surjective homomorphism from $\mathbb{Z}$ to a field F. Prove that F is isomorphic to $\mathbb{Z}_p$ for some prime p.