1. **(20 points)** Fill in the following table. You need not justify your results. In each cell, place either an X or a checkmark (✓). Place an X if the ring named in the column is not an algebra of the type named in the row, and a checkmark if the ring named in the column is an algebra of the type named in the row. Place one of these two marks in *every* cell of the table; empty cells will be automatically incorrect.

	$\mathbb{R}$	$\mathbb{Z}[x]$	$\mathbb{Z}[\sqrt{2}, \sqrt[4]{2}, \sqrt[8]{2}, \ldots]$	$\mathbb{Z}_7[x]$	$\mathbb{Q}[x]/\langle x^2 \rangle$
Principal ideal domain (PID)					
Euclidean domain					
Field					
Unique factorization domain (UFD)					

2. **(20 points)** If R is a principal ideal domain, and I is a proper ideal of R (that is, an ideal smaller than the entirety of R), prove that I is contained (possibly nonstrictly) in a maximal ideal of R.

3. **(20 points)** Let $f(x) \in \mathbb{Z}[x]$ be given by the formula $a_n x^n + \cdots + a_1 x + a_0$. Prove that if $\frac{p}{q} \in \mathbb{Q}$ is a fraction in lowest terms such that $f(\frac{p}{q}) = 0$, then $p \mid a_0$ and $q \mid a_n$. (This result is known as the *Rational Root Theorem*.)

4. (a) **(10 points)** Prove that $\mathbb{Z}_2[x]/\langle x^3 + x^2 + 1 \rangle$ is a field. What is its order?

   (b) **(10 points)** Prove that $\mathbb{Z}_7[x]/\langle 3x^2 + x + 4 \rangle$ is *not* a field. Is it an integral domain?

5. **(20 points)** Prove that every prime element of an integral domain is irreducible.

6. **(Bonus question, 10 extra points)** Prove that every irreducible element of a principal ideal domain is prime.