1. **(40 points)** Fill in the following table. You need not justify your results. In each cell, place either an X or a checkmark. Place a checkmark if a structure of the type named in the row *is always* a structure of the type named in the column, and an X if not. Fill in every cell. The tautological cells are pre-filled. To round off the values, you get 2 points just for being you.

	is always...					
	a commutative ring	an integral domain	a PID	a UFD	a Noetherian domain	a field
An integral domain		✓				
A principal ideal domain			✓			
A unique factorization domain				✓		
A Euclidean domain						
A field						✓
$F[x]$, where F is a field						
$D[x]$, where D is a PID						

2. **(20 points)** Prove the following statements about rings:

   (a) **(10 points)** In a ring R (which may not be commutative or possess a unity) prove that for any $a \in R$, $a \cdot 0 = 0$.

   (b) **(10 points)** In a ring R with unity, prove that for any $a \in R$, $-a = -1 \cdot a$.

3. **(20 points+10 bonus)** Let D be an integral domain, and let $a, b \in D$.

   (a) **(20 points)** Prove that if $a^5 = b^5$ and $a^3 = b^3$, then $a = b$.

   (b) **(10 points)** Find (with proof) the best possible conditions on positive m and n such that if $a^m = b^m$ and $a^n = b^n$, then $a = b$.

4. **(20 points)** Let R be the ring of all functions from $\mathbb{Q}$ to $\mathbb{Q}$, with the operations of multipliction and addition being $(f+g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x) \cdot g(x)$, and let $I = \{f(x) : f(0) = 0\}$.

   (a) Prove that I is an ideal.

   (b) Prove that I is a maximal ideal.

5. (a) **(10 points)** Determine whether $\mathbb{Q}[x]/\langle x^4 + 3x^2 + 3\rangle$ is a field.

   (b) **(10 points)** Find the order of the ring $\mathbb{Z}_3[x]/\langle x^3 + 2x^2 + 1\rangle$, and determine whether it is a field.

6. **(20 points)** Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree n, and let E be the splitting field of $f(x)$ over $\mathbb{Q}$.

   (a) **(10 points)** Prove that $[E : \mathbb{Q}]$ can be written as a product $k_1 k_2 k_3 \cdots k_n$, where each k_i is a positive integer less than or equal to i (hint: use induction on n).

   (b) **(10 points)** Prove that if $f(x)$ is irreducible, then $[E : \mathbb{Q}]$ is divisible by n.

   (c) **(10 point bonus)** The results above indicate that if $f(x)$ is an irreducible cubic, $[E : \mathbb{Q}]$ is 3 or 6. Are both possible? Prove or give examples.