1. **(10 points)** *Let* $f(x) = 3x^2 - 5x - 2$.

   (a) **(6 points)** *At which values of* x *does* $f(x) = 0$*?*

   The quadratic equation $3x^2 - 5x - 2 = 0$ can be solved by factoring, by completion of the square, or by the quadratic formula. The first and third techniques are the simplest. We might observe that $3x^2 - 5x - 2 = (3x+1)(x-2)$, so either $3x + 1 = 0$ or $x - 2 = 0$, yielding $x = \frac{-1}{3}$ or $x = 2$.

   Alternatively, we could use the quadratic formula to determine that

   $$x = \frac{5 \pm \sqrt{(-5)^2 - 4 \cdot 3(-2)}}{2 \cdot 3} = \frac{5 \pm \sqrt{49}}{6} = \frac{5 \pm 7}{6}$$

   so either $x = \frac{12}{6} = 2$ or $x = \frac{-2}{6} = \frac{-1}{3}$.

   (b) **(4 points)** *What is the average rate of change of the function* $f(x)$ *between the values* $x = -2$ *and* $x = 0$*?*

   The average value is given by the quotient

   $$\frac{f(0) - q(-2)}{0 - (-2)} = \frac{(3 \cdot 0^2 - 5 \cdot 0 - 2) - (3(-2)^2 - 5(-2) - 2}{2} = \frac{-2 - 20}{2} = \frac{-22}{2} = -11$$

2. **(15 points)** *I want to make five liters of a 60% sucrose solution; I have large quantities of 30% sucrose and 70% sucrose. How much should I blend from of these two solutions to get the desired mixture?*

   Let's call the quantity in liters of 30% solution x, so that the quantity of 70% solution will be $5 - x$. We can lay the information we have out in a table:

	Volume	Concentration	Sucrose volume
30%	x	0.3	
70%	$5 - x$	0.7	
Combined	5	0.6	

   Noting that the volume of sucrose is the product of the total volume and the concentration, so we can populate the rest of the table to get the following:

	Volume	Concentration	Sucrose volume
30%	x	0.3	$0.3x$
70%	$5 - x$	0.7	$0.7(5 - x)$
Combined	5	0.6	3

   SO to get the correct total quantity of sucrose when we combine the two liquids, we want it to be the case that

   $$0.3x + 0.7(5 - x) = 3$$

   which simplifies to

   $$3.5 - 0.4x = 3$$

   with the result that $x = \frac{0.5}{0.4} = 1.25$, so we want 1.25 liters of 30% solution, and 3.75 liters of the 70% solution.

3. **(15 points)** *Perform the following arithmetic and algebraic operations.*

   (a) **(5 points)** *Simplify the expression* $\frac{(x^2y)^3}{x(y^2z)^2}$.
   Using exponential distribution-and-gathering techniques:

$$\frac{(x^2y)^3}{x(y^2z)^2} = \frac{x^6y^3}{xy^4z^2} = \frac{x^5}{yz^2}.$$

   One might alternatively express the final answer as $x^5y^{-1}z^{-2}$.

   (b) **(6 points)** *Simplify the rational expression* $\frac{x+1}{x^2-2} - \frac{2}{x-1}$.
   Finding a common denominator and using straightforward algebraic techniques:

$$\begin{aligned}\frac{x+1}{x^2-2} - \frac{2}{x-1} &= \frac{(x+1)(x-1)}{(x^2-2)(x-1)} - \frac{(x^2-2)2}{(x^2-2)(x-1)} \\ &= \frac{(x+1)(x-1)-(x^2-2)2}{(x^2-2)(x-1)} \\ &= \frac{x^2-1-2x^2+4}{x^3-x^2-2x+2} \\ &= \frac{3-x^2}{x^3-x^2-2x+2}\end{aligned}$$

   (c) **(4 points)** *Calculate* $(-8)^{5/3}$.
   Exploding the rational exponent into two individually calculable steps, we get:

$$(-8)^{5/3} = \left((-8)^{1/3}\right)^5 = \sqrt[3]{-8}^5 = (-2)^5 = -32$$

4. **(10 points)** *Linh is in a class where the grade is determined based solely on exams, and where the final exam is worth twice as much as the three midterms. If Linh got grades of 75%, 82%, and 71% on the three midterms, what grade will she need on the final exam to get an 80% overall in the class?*

   Let x represent the final exam grade which will give her an overall average of 80%. The final grade is the average of 5 things: each of the three midterms, and the final twice over, so it should be the case that $\frac{75+82+71+x+x}{5} = 80$. This is a straightforwardly solved linear equation:

$$\begin{aligned}75+82+71+2x &= 5\cdot 80 \\ 228+2x &= 400 \\ 2x &= 172 \\ x &= 86\end{aligned}$$

   so she will need a score of 86% on the final.

5. **(23 points)** *Answer the following questions about the functions* $f(x) = x^3 - 8$ *and* $g(x) = \sqrt{x-3}$. *In each question asking for multiple answers,* label which is which.

   (a) **(5 points)** *Find the inverse of the function* $f(x)$.

We know that $f(f^{-1}(x)) = x$, and by expanding $f(x)$ we can express $f^{-1}(x)$ in terms of x:

$$\begin{aligned} x &= f(f^{-1}(x)) \\ x &= \left(f^{-1}(x)\right)^3 - 8 \\ x &= \left(f^{-1}(x)\right)^3 - 8 \\ x + 8 &= \left(f^{-1}(x)\right)^3 \\ \sqrt[3]{x+8} &= f^{-1}(x) \end{aligned}$$

(b) **(3 points)** *Write formulas, which need not be simplified, for $(g - f)(x)$ and $\frac{f}{g}(x)$.*

$$\begin{aligned} (g - f)(x) = g(x) - f(x) &= \sqrt{x-3} - (x^3 - 8) \\ \frac{f}{g}(x) = \frac{f(x)}{g(x)} &= \frac{x^3 - 8}{x - 3} \end{aligned}$$

(c) **(4 points)** *Write formulas, which need not be simplified, for $f(g(x))$ and $f(f(x))$.*

$$\begin{aligned} f(g(x)) = f\left(\sqrt{x-3}\right) &= (x-3)^{3/2} - 8 \\ f(f(x)) = f\left(x^3 - 8\right) &= (x^3 - 8)^3 - 8 \end{aligned}$$

(d) **(5 points)** *Determine the domains of $f(x)$ and $g(x)$.*

Since $f(x)$ is a polynomial, we know that the domain of $f(x)$ consists of all real numbers, which could be written out in exactly those terms or described as the interval $(-\infty, \infty)$.

On the other hand, $g(x)$ includes a square root, so its domain must be limited to those values where the parameter of that square root is non-negative. Thus, our domain is limited to those values where $x - 3 \geq 0$, or where $x \geq 3$; written in interval form, this would be $[3, \infty)$.

(e) **(6 points)** *Determine the domains of $(f + g)(x)$, $(f - g)(x)$, $(fg)(x)$, $\frac{f}{g}(x)$, and $\frac{g}{f}(x)$.*

We recall that $f + g$, $f - g$, and fg have domains given by the overlap of the domains of f and g, so all three of these functions have domains given by $x \geq 3$ and $x \neq -2$; if desired, this can be expressed as the interval form $[3, \infty)$.

However, $\frac{f}{g}$ has an additional restriction: in addition to needing x-values in the domain of f and g, $\frac{f}{g}$ is only defined where $g(x) \neq 0$. Since $g(x) = \sqrt{x-3}$, it is clear that $g(x) = 0$ when $x - 3 = 0$, so we must exclude 3 from the domain of $\frac{f}{g}$; thus $\frac{f}{g}$ has a domain given by the condition $x > 3$, or in interval form, $(3, \infty)$.

Likewise, $\frac{g}{f}$ is only defined where $f(x) \neq 0$. Since $f(x) = x^3 - 8$, it is clear that $f(x) = 0$ when $x = \sqrt[3]{8} = 2$, so we must exclude 2 from the domain of $\frac{g}{f}$, but it was, in fact, already excluded; thus $\frac{g}{f}$ has a domain given by the condition $x \geq 3$, or in interval form, $[3, \infty)$.

6. **(9 points)** *A bathtub holds 50 gallons of water and is currently filled. The drain is opened, causing water to flow out at a constant rate, and 3 minutes later, the bathtub only contains 14 gallons of water.*

(a) **(5 points)** *Write a function $f(t)$ to represent the volume of water still in the tub t minutes after opening the drain.*

We know this is a linear function, since the rate of outflow is constant. We further are told in the problem statement that $f(0) = 50$ and $f(3) = 14$. The graph of such a function is thus a line through $(0, 50)$ and $(3, 14)$, and we can easily find the equation for such a line by computing the slope

$$m = \frac{14 - 50}{3 - 0} = -12$$

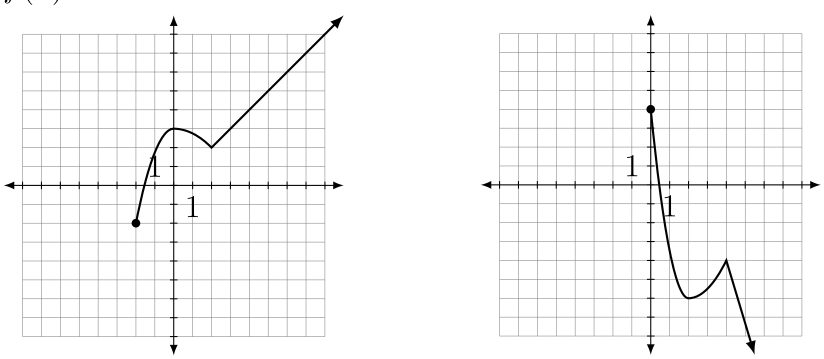
and putting the equation in the slope-intercept form $y = -12x + 50$; thus the function required is $f(t) = 50 - 12t$.

(b) **(4 points)** *How many minutes will it take the tub to empty completely?*

Since $f(t)$ represents the volume of water after t minutes, we want to know when $f(t)$ will be zero. Using the formual above, we thous want to solve the linear equation $50 - 12t = 0$, whose solution is $t = \frac{50}{12} = \frac{25}{6}$, so it takes $\frac{25}{6}$ minutes to drain (which could alternatively be written as $4\frac{1}{6}$ minutes, or 4 minutes 10 seconds).

7. **(13 points)** *Answer the following questions about functions and their graphs.*

(a) **(5 points)** *A function $f(x)$ is shown on the graph to the left, and a transformation $g(x)$ of this function is shown on the right. Find a formula for $g(x)$ in terms of the formula for $f(x)$.*



The transformation includes a vertical flip and stretch by a factor of 2 (the magnitude of the stretch can be computed to be 2 by noticing that the original vertical distance between the endpoint at height -2 and the local maximum at height 3 is transformed into an endpoint at height 4 and a local minimum at heigh -6), as well as a horizontal translation to the right by 2 units. Thus $g(x) = -2f(x - 2)$.

(b) **(3 points)** *For* $f(x) = \begin{cases} x^2 - 3 \text{ if } & x < 4 \\ \sqrt{x} \text{ if } & x \geq 4 \end{cases}$, *calculate the following values:*

- $f(1)$. Since $1 < 4$, $f(1) = 1^2 - 3 = -2$.
- $f(4)$. Since $4 \geq 4$, $f(4) = \sqrt{4} = 2$.
- $f(9)$. Since $9 \geq 4$, $f(9) = \sqrt{9} = 3$.

(c) **(5 points)** *Determine the equation of the line through the points* $(2, 1)$ *and* $(8, 3)$.

We start by calculating the slope of the line:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 1}{8 - 2} = \frac{2}{6} = \frac{1}{3}$$

and then, using one of the points — in this example $(2,1)$, although the choice of point doesn't matter — to get the equation of the line in point-slope form:

$$(y-1)=\frac{1}{3}(x-2)$$

which can, if desired, be algebraically rearranged to be in slope-intercept form:

$$y=\frac{1}{3}x+\frac{1}{3}$$