1. **(15 points)** *Answer the following questions about growth and decay.*

   (a) **(5 points)** *A metal ingot is removed from a furnace into a warm metalworking studio; its temperature t minutes after removal is given by the function* $f(t) = 90 + 410e^{-0.05t}$. *How long will it take the ingot to cool to* $200°F$*?*

   We want a solution to the exponential equation $90 + 410e^{-0.05t} = 200$. Performing appropriate arithmetic on both sides of the equality, we can eventually liberate the value of t:

$$\begin{aligned} 90 + 410e^{-0.05t} &= 200 \\ 410e^{-0.05t} &= 110 \\ e^{-0.05t} &= \frac{110}{410} \\ \ln e^{-0.05t} &= \ln \frac{110}{410} \\ -0.05t &= \ln \frac{11}{41} \\ t &= \frac{\ln \frac{11}{41}}{-0.05} \end{aligned}$$

   For reference, this value is about 26 minutes, although you could not reasonably determine that without a calculator.

   (b) **(4 points)** *You have \$500 invested in an account which bears 2% annual interest, compounding quarterly. Produce a function describing the value in your account after t years.*

   After t years, your account will have earned $\frac{2\%}{4}$ interest $4t$ times, for a total value of $f(t) = 500\left(1 + \frac{2\%}{4}\right)^{4t} = 500(1.005)^{4t}$.

   (c) **(6 points)** *You hope to cash the account from the previous question out when it reaches a balance of \$600. Based on the function from the previous question, how many years do you need to wait until you can do so?*

   We want a solution to the equation $500(1.005)^{4t} = 600$. Performing appropriate arithmetic on both sides of the equality, we can eventually liberate the value of t:

$$\begin{aligned} 500(1.005)^{4t} &= 600 \\ (1.005)^{4t} &= \frac{600}{500} \\ \ln(1.005^{4t}) &= \ln \frac{6}{5} \\ 4t \ln 1.005 &= \ln \frac{6}{5} \\ t &= \frac{\ln \frac{6}{5}}{4 \ln 1.005} \end{aligned}$$

   For reference, this answer is about nine years.

2. **(20 points)** *Solve the inequality* $\frac{s+3}{(s-1)^2} \leq \frac{1}{s+2}$.

We will move the entire expression to the left side of the inequality, and make a single fraction with a common denominator and a factored numerator:

$$\frac{s+3}{(s-1)^2} \leq \frac{1}{s+2}$$
$$\frac{s+3}{(s-1)^2} - \frac{1}{s+2} \leq 0$$
$$\frac{(s+3)(s+2) - 1(s-1)^2}{(s-1)^2(s+2)} \leq 0$$
$$\frac{(s^2+5s+6)) - (s^2-2s+1)}{(s-1)^2(s+2)} \leq 0$$
$$\frac{7s+5}{(s-1)^2(s+2)} \leq 0$$

We are interested, then, in when $\frac{7s+5}{(s-1)^2(s+2)}$ is nonpositive. It clearly has the potential for sign transition when either the numerator is zero (at $s = \frac{-5}{7}$) or when the denominator is zero (at $s = 1$ with multiplicity 2 and $s = -2$). The expression as a whole has long-term behavior as $\frac{7s}{s^3}$, and will be positive when x is very large in either direction, so we know this expression is positive on the interval $(-\infty, -2)$, negative on $(-2, \frac{-5}{7})$, positive on $(\frac{-5}{7}, 1)$, and positive again on $(1, \infty)$, noting the double zero with no sign change. Thus the interval where it is negative is $(-2, \frac{-5}{7})$; but we need to include the place where the expression is zero because the inequality is nonstrict, and so we get $(-2, \frac{-5}{7}]$ as our answer.

3. **(20 points)** *Answer the following questions preparatory to sketching the rational function* $h(x) = \frac{-5(x+2)(x-3)}{(x-1)^2}$.

   (a) **(4 points)** *What is the function's domain?*

   Since the denominator is zero when $x = 1$, our domain is where $x \neq 1$. If put in interval form, this is $(-\infty, 1) \cup (1, \infty)$.

   (b) **(5 points)** *Does this function have x-intercepts, and if so, what are they?*

   The numerator is zero when $x = -2$ or $x = 3$; both of these are within the domain, so they are both places where $h(x) = 0$. Thus they are x-intercepts.

   (c) **(4 points)** *Where, if anywhere, are this function's vertical asymptotes?*

   The denominator is zero and the numerator nonzero at $x = 1$, so $x = 1$ is a vertical asymptote of this function.

   (d) **(5 points)** *How does this function behave as x becomes very large? How does it behave as x becomes very highly negative? Label which is which.*

   If we expand the numerator, the function could be written as $h(x) = \frac{-5x^2-x-6}{x^2-2x+1}$. For x of very large magnitude, the dominant terms in the numerator and denominator make this approximately $\frac{-5x^2}{x^2} = -5$. As $x \to +\infty$, this expression will thus approach -5, and, as $x \to -\infty$, this expression will still approach -5.

   (e) **(2 point)** *Does this function have a maximum or minimum value? Why or why not?*

   It has no maximum because the asymptotic behavior induces arbitrarily-large magnitude positive and values close to the asymptote (which has multiplicity 2 so it only flies off the graph in one direction). It *does* have a minimum, which may be surprising, but that occurs because there is no obstruction to its having a minimum.

Its minimum actually occurs at $(13, \frac{-125}{24})$, but calculating that is very difficult (wait until MATH 205!).

4. **(15 points)** *Answer the following questions about the quadratic* $q(x) = -2x^2 + 12x - 20$.

   (a) **(8 points)** *Put the quadratic* $q(x)$ *in standard form.*

$$\begin{aligned} q(x) &= -2x^2 + 12x - 20 \\ &= -2(x^2 - 6x) - 20 \\ &= -2(x^2 - 6x + 9 - 9) - 20 \\ &= -2(x^2 - 6x + 9) + 18 - 20 \\ &= -2(x-3)^2 - 2 \end{aligned}$$

   (b) **(2 point)** *Does* $q(x)$ *have a maximum or minimum value? If so, identify which it is and what its value is.*

   It obtains a maximum (since it is a quadratic with negative x^2 coefficient) at its vertex, the point $(3, -2)$.

   (c) **(5 points)** *Determine the* x*-intercepts of this quadratic if they exist (explicitly stating if they do not exist), and its* y*-intercept. Label which is which.*

   This quadratic has no x-intercepts. This can be determined either by noticing that its maximum y-value, stated above, is -2 so this function never gets as high as the x-axis. Alternatively, one can use the quadratic formula to discover that the zeroes $\frac{-12 \pm \sqrt{144-160}}{-4}$ are complex.

   The y-intercept is $q(0) = -20$.

5. **(15 points)** *Answer the following questions concerning logarithms.*

   (a) **(5 points)** *Solve for* x *in the exponential equation* $5 \cdot 2^{(x^2-4x-1)} = 80$.

   Dividing both sides by 5, we see that $2^{(x^2-4x-1)} = 16$. Since $16 = 2^4$, the exponent there must be 4, so $x^2 - 4x - 1$ must equal 4. Thus $x^2 - 4x - 5 = 0$, which can be solved with either the quadratic formula or factorization to have solutions $x = -1$ and $x = 5$.

   (b) **(6 points)** *Find the domain of the function* $g(x) = \frac{\ln(15-2x)}{\sqrt{x-5}-1}$.

   There are in fact three restrictions on this function: first, the parameter of the square root must be non-negative, second, the parameter of the logarithm must be positive, and third, the denominator of the fraction must be nonzero. Thus, we require that $x - 5 \geq 0$, $15 - 2x > 0$, and $\sqrt{x-5} - 1 \neq 0$. The first and second requirements are easily simplified to $x \geq 5$ and $x < \frac{15}{2}$. The last requirement is that $\sqrt{x-5} \neq 1$ so $x - 5 \neq 1$ and so $x \neq 6$. Thus, our domain is $5 \leq x < \frac{15}{2}$ with $x \neq 6$, or, in interval form, $[5, 6) \cup (6, \frac{15}{2})$.

   (c) **(4 points)** *Determine the value of* $2\log_3 6 + \log_3 12 - 4\log_3 2$.

   Using the various logarithm rules:

$$2\log_3 6 + \log_3 12 - 4\log_3 2 = \log_3 6^2 + \log_3 12 - \log_3 2^4 = \log_3 \frac{6^2 \cdot 12}{2^4} = \log_3 27 = 3$$

6. **(15 points)** *Answer the following questions about polynomial functions.*

   (a) **(3 points)** *Identify all the* potential *rational roots of* $2x^4 - 3x^3 + x^2 - 6$. *Do not check which are actual roots.*

   By the Rational Root Theorem, the possible numerators are 1, 2, 3, and 6, and the denominators 1 and 2. Since any combination can appear with either sign, there are 12 potential roots after winnowing out duplications: $1, -1, 2, -2, 3, -3, 4, -4, \frac{1}{2}, \frac{-1}{2}, \frac{3}{2}, \frac{-3}{2}$.

   Incidentally, -1 is a root of this polynomial, although you were not asked to determine that. It also has two complex roots and one irrational real root.

   (b) **(7 points)** *Identify the x-intercepts, y-intercept, and long-term behavior of the polynomial $f(x) = 4(x+1)^3(x-3)$.*

   The y-intercept is $f(0) = 4(0+1)^3(0-3) = -12$; if you prefer to name a point in the plane, it may be denoted $(0, -12)$.

   The x-intercepts occur when $4(x+1)^3(x-3) = 0$, which are, based on when each factor is zero, the points $x = -1$ and $x = 3$, the former with a multiplicity of 3. If you prefer to name points in the plane, they may be denoted $(-1, 0)$ and $(3, 0)$.

   The leading term of the polynomial is $4x^4$; this is a monomial of even degree (4) and positive coefficient (also 4), so as $x \to +\infty$, $f(x) \to +\infty$, and as $x \to -\infty$, $f(x) \to +\infty$.

   (c) **(5 points)** *Using either synthetic or long division, find the quotient and remainder of the operation $\frac{x^4+8x^2-2x+7}{x+3}$.*

   Here is the result of a long division, with a quotient of $x^3 - 3x^2 + 17x - 53$ and a remainder of 166:

$$
\begin{array}{r|rrrrr}
 & x^3 & -3x^2 & +17x & -53 & \\
\hline
x+3 & x^4 & & +8x^2 & -2x & +7 \\
 & -x^4 & -3x^3 & & & \\
\cline{2-3}
 & & -3x^3 & +8x^2 & & \\
 & & 3x^3 & +9x^2 & & \\
\cline{3-4}
 & & & 17x^2 & -2x & \\
 & & & -17x^2 & -51x & \\
\cline{4-5}
 & & & & -53x & +7 \\
 & & & & 53x & +159 \\
\cline{5-6}
 & & & & & 166
\end{array}
$$

   Alternatively, we could perform synthetic division:

$$
\begin{array}{r|rrrrr}
-3 & 1 & 0 & 8 & -2 & 7 \\
 & & -3 & 9 & -51 & 159 \\
\hline
 & 1 & -3 & 17 & -53 & 166
\end{array}
$$

   and the first three entries of the lowest row give us the coefficients in the quotient $x^3 - 3x^2 + 17x - 53$, while the last entry is the remainder 166.