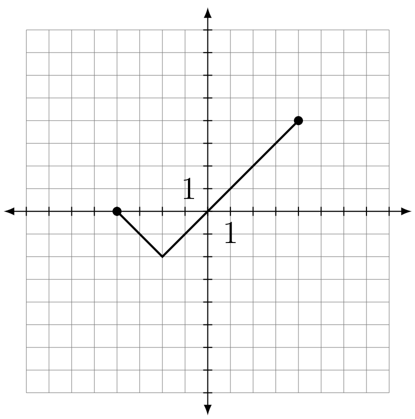
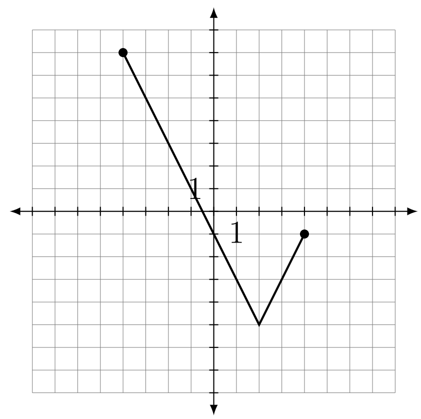
1. **(10 points)** *Answer the following questions about graphs.*

   (a) **(3 points)** *Given the piecewise function* $g(x) = \begin{cases} x^2 - 2 & \text{if} \quad x > 1 \\ 2x - 5 & \text{if} \quad x \leq 1 \end{cases}$, *calculate the following values:*

   - $g(3)$.
     Since $3 > 1$, $g(3) = 3^2 - 2 = 7$.
   - $g(2)$.
     Since $2 > 1$, $g(2) = 2^2 - 2 = 2$.
   - $g(1)$.
     Since $1 \leq 1$, $g(1) = 2 \cdot 1 - 5 = -3$.

   (b) **(4 points)** *For* $f(x)$ *as shown on the graph, sketch the graph of its transformation* $g(x) = 2f(-x) - 1$.





   The transformation includes, in order, a horizontal flip (due to the appearance of $f(-x)$), a vertical stretch by a factor of 2 (since the function result is multiplied by 2) and a shift downwards by 1 (since 1 is subtracted from the function value). We see what this looks like above.

   (c) **(3 points)** *Determine the equation of the line through the points* $(1, 4)$ *and* $(4, -5)$.
   We start by calculating the slope of the line:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-5 - 4}{4 - 1} = \frac{-9}{3} = -3$$

   and then, using one of the points — in this example $(1, 4)$, although the choice of point doesn't matter — to get the equation of the line in point-slope form:

$$(y - 4) = -3(x - 1)$$

   which can, if desired, be algebraically rearranged to be in slope-intercept form:

$$y = -3x + 7$$

2. **(6 points)** *The Hong Kong Cavaliers have 1000 fans who would come to a concert if tickets cost \$10. Polling of the fanbase suggests that every increase in the ticket price by \$1 would reduce attendance by 50.*

(a) **(4 points)** *Express the number of concertgoers as a function of the ticket price.*

Let us denote the ticket price by x; we have a base attendance of 1000, and we diminish that by 50 for eachdollar by which x exceeds 10; in otherwise, we must take $50(x-10)$ people away from that base attendance to determine the attendance at a ticket price x. Thus our demand function is $D(x) = 1000 - 50(x-10)$. This may, but need not, be algebraically simplified to $D(x) = 1500 - 50x$.

(b) **(2 points)** *Express the total revenue from ticket sales as a function of the ticket price.*

As seen above, we sell $1500 - 50x$ tickets at ticket price x; our total revenue from so doing would be the product of the demand and the price, or $R(x) = D(x) \cdot x = (1500 - 50x)x$. This may, but need not, be expanded algebraically to give $R(x) = -50x^2 + 1500x$.

3. **(4 points)** *Alice is 30 years old and her son is currently three years old; she notes with interest that her age is currently ten times his, and would like to know how many years it will be until she is only four times as old as he is; algebraically determine how long it will take.*

Let x represent the number of years until the desired event. At that time Alice will be $30 + x$ years old and her son $3 + x$. The first quantity should be four times the second, so we require that $30 + x = 4(3 + x)$, which we can solve as such:

$$\begin{aligned} 30 + x &= 4(3 + x) \\ 30 + x &= 12 + 4x \\ 18 &= 3x \\ 6 &= x \end{aligned}$$

so in six years Alice will be four times as old as her son. We can verify this by plugging the value in: Alice will then be 36 and her son 9, and 36 is indeed $4 \cdot 9$.

4. **(15 points)** *Answer the following questions about the functions* $f(x) = \frac{2}{x+1}$ *and* $g(x) = \frac{x}{x+2}$. *In each question asking for multiple answers,* label which is which.

(a) **(3 points)** *Find the inverse of the function* $f(x)$.

We know that $f(f^{-1}(x)) = x$, and by expanding $f(x)$ we can express $f^{-1}(x)$ in terms of x:

$$\begin{aligned} x &= f(f^{-1}(x)) \\ x &= \frac{2}{f^{-1}(x) + 1} \\ (f^{-1}(x) + 1)x &= 2 \\ f^{-1}(x) + 1 &= \frac{2}{x} \\ f^{-1}(x) &= \frac{2}{x} - 1 \end{aligned}$$

(b) **(2 points)** *Write formulas, which need not be simplified, for* $(g - f)(x)$ *and* $\frac{f}{g}(x)$.

$$(g - f)(x) = g(x) - f(x) = \frac{x}{x+2} - \frac{2}{x+1}$$

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{\frac{2}{x+1}}{\frac{x}{x+2}}$$

(c) **(3 points)** *Write formulas, which need not be simplified, for $f(g(x))$ and $f(f(x))$.*

$$f(g(x)) = f\left(\frac{x}{x+2}\right) = \frac{2}{\frac{x}{x+2}+1}$$
$$f(f(x)) = f\left(\frac{2}{x+1}\right) = \frac{2}{\frac{2}{x+1}+1}$$

(d) **(3 points)** *Determine the domains of $f(x)$ and $g(x)$.*

Since $f(x)$ has a denominator of $x+2$ (but is otherwise free of "dangerous" calculations), we know that the domain of $f(x)$ consists of those points where $x+2 \neq 0$, or where $x \neq -2$; if desired, this can be expressed as the interval form $(-\infty, -2) \cup (-2, \infty)$.

Likewise, $g(x)$ has a denominator of $x+1$, but is otherwise free of dangerous calculations, so we know that the domain of $g(x)$ consists of those points where $x+1 \neq 0$, or where $x \neq -1$; if desired, this can be expressed as the interval form $(-\infty, -1) \cup (-1, \infty)$.

(e) **(4 points)** *Determine the domains of $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$, and $\frac{f}{g}(x)$.*

We recall that $f+g$, $f-g$, and fg have domains given by the overlap of the domains of f and g, so all three of these functions have domains given by $x \neq -1$ and $x \neq -2$; if desired, this can be expressed as the interval form $(-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$.

However, $\frac{f}{g}$ has an additional restriction: in addition to needing x-values in the domain of f and g, $\frac{f}{g}$ is only defined where $g(x) \neq 0$. Since $g(x) = \frac{x}{x+2}$, it is clear that $g(x) = 0$ when $x = 0$, so we must exclude 0 from the domain of $\frac{f}{g}$; thus $\frac{f}{g}$ has a domain given by the conditions $x \neq 0$, $x \neq -1$, and $x \neq -2$, or in interal form, $(-\infty, -2) \cup (-2, -1) \cup (-1, 0) \cup (0, \infty)$.

5. **(15 points)** *Perform the following arithmetic and algebraic operations.*

(a) **(3 points)** *Factor the quadratic $x^2 - 8x + 12$.*

Trial and error suggests the possibile factorizations $(x-1)(x-12)$, $(x-2)(x-6)$, $(x-3)(x-4)$, $(x+1)(x+12)$, $(x+2)(x+6)$, and $(x+3)(x+4)$. The second of these is correct. This can also be found using the quadratic formula to note that $x^2 - 8x + 12$ is zero when $x = 2$ and $x = 6$.

(b) **(3 points)** *Expand and simplify the polynomial $(x^3 + 5) - (2x - 1)(x^2 + 3x)$.*

Using straightforward algebraic techniques:

$$\begin{aligned}(x^3+5) - (2x-1)(x^2+3x) &= (x^3+5) - (2x^3 + 6x^2 - x^2 - 3x) \\ &= -x^3 - 5x^2 + 3x + 5\end{aligned}$$

(c) **(3 points)** *Calculate $16^{3/4}$.*

Exploding the rational exponent into two individually calculable steps, we get:

$$16^{3/4} = \left(16^{1/4}\right)^3 = \sqrt[4]{16}^3 = 2^3 = 8$$

(d) **(3 points)** *Simplify the expression* $\frac{x}{x-4} - \frac{3}{x+6}$.
We find a common denominator and simplify the numerator:

$$\begin{aligned}\frac{x}{x-4} - \frac{3}{x+6} &= \frac{x(x+6)}{(x-4)(x+6)} - \frac{(x-4)3}{(x-4)(x+6)} \\ &= \frac{x(x+6)-(x-4)3}{(x-4)(x+6)} \\ &= \frac{x^2+6x-(3x-12)}{x^2+2x-24} \\ &= \frac{x^2+3x+12}{x^2+2x-24}\end{aligned}$$

The denominator could be left factored; expansion versus factorization is a stylistic variation within the bounds of what is considered "simplified".

(e) **(3 points)** *Simplify the expression* $\frac{(2x^3)^2(3x^4)}{(x^3)^4}$.
Using exponential distribution-and-gathering techniques:

$$\frac{(2x^3)^2(3x^4)}{(x^3)^4} = \frac{2^2x^6 \cdot 3x^4}{x^{12}} = \frac{12x^{10}}{x^{12}} = \frac{12}{x^2}$$

One might alternatively express the final answer as $12x^{-2}$.

6. **(10 points)** *Answer the following questions about the quadratic* $q(x) = 6x^2 + 12x - 5$.

(a) **(2 points)** *What is the average rate of change of the function* $q(x)$ *between the points* $x = -1$ *and* $x = 1$?
The average value is given by the quotient

$$\frac{q(1)-q(-1)}{1-(-1)} = \frac{(6 \cdot 1^2 + 12 \cdot 1 - 5) - (6(-1)^2 + 12(-1) - 5)}{2} = \frac{13-(-11)}{2} = \frac{24}{2} = 12$$

(b) **(3 points)** *Determine the* x*-intercepts of this quadratic if they exist (explicitly stating if they do not exist), and its* y*-intercept. Label which is which.*
The y-intercept is simply $q(0) = 6 \cdot 0^2 + 12 \cdot 0 - 5 = -5$. The x-intercepts are given by finding the solutions to the equation $q(x) = 0$; since this equation is $6x^2 + 12x - 5 = 0$, it is easily solved using the quadratic equation:

$$x = \frac{-12 \pm \sqrt{12^2 - 4 \cdot 6(-5)}}{2 \cdot 6} = \frac{-12 \pm \sqrt{264}}{12}$$

This can, but need not, be simplified to $1 \pm \frac{\sqrt{66}}{6}$.

7. **(6 points)** *The moon is about 240,000 miles from the earth. A rocket going to the moon goes a constant speed after launch, and then fires a secondary booster 40,000 miles into the trip to increase its speed by 8000 mph. If we want the rocket to reach the moon 30 hours after launch, how quickly does it need to be going initially?*

Let's call the launch speed in miles per hour x, so that its second-stage speed is $x + 8000$ mph. We know the first leg of the journey is 40,000 miles and the second leg 200,000 and the trip as a whole takes 30 hours. We can lay this information out in a table:

	Time	Speed	Distance
First stage		x mph	40000 miles
Second stage		$x + 8000$ mph	200000 miles
Overall	30 hours		240000 miles

Note that distances in each row are products of times and speeds, and that the overall distance and time is the sum of the distances and times respectively for the two legs of the journey (note the same is *not* true of speeds!). On this basis we can populate the rest of the table to get the following:

	Time	Speed	Distance
First stage	$\frac{40000}{x}$ hours	x mph	40000 miles
Second stage	$\frac{200000}{x+8000}$	$x + 8000$ mph	200000 miles
Overall	30 hours		240000 miles

Since the total travel time is 30 hours, we thus know that

$$\frac{40000}{x} + \frac{200000}{x + 8000} = 30$$

which, multiplied by the common denominator, becomes

$$40000(x + 8000) + 200000x = 30x(x + 8000)$$

which simplifies to

$$240000x + 320000000 = x^2 + 240000x$$

so $x^2 = 320000000$. Then $x = \pm\sqrt{320000000} = \pm 4000\sqrt{20}$. The negative value here is clearly nonsense, so we want an initial speed of $4000\sqrt{20} \approx 17900$ miles per hour.