1. **(10 points)** *Answer the following questions about trigonometry.*

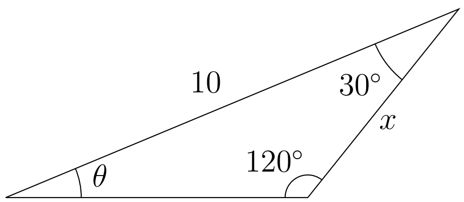
   (a) **(3 points)** *Find the terminal point associated with* $t = \frac{-9\pi}{2}$.

   (b) **(4 points)** *Identify the period, amplitude, and vertical shift of the periodic function* $g(x) = 7\sin(5x) - 2$.

   (c) **(3 points)** *Evaluate* $\cot\frac{5\pi}{6}$.

2. **(10 points)** *Calculate the labeled quantities in the triangles (not drawn to scale) below.*

   (a) **(5 points)** *Determine* x *and* θ*:*



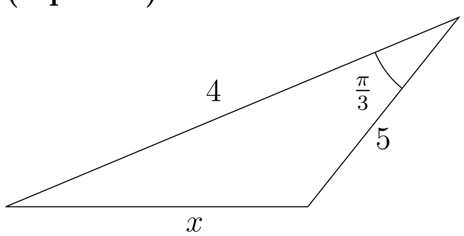
Since the angles add up to 180°, we know that $120 + 30 + \theta = 180$, so $\theta = 30°$.

In this triangle two angles and only one side have been specified, making it a prime candidate for application of the Rule of Sines. Looking at the two labeled sides and their opposite angles, we get:

$$\frac{x}{\sin\theta} = \frac{10}{\sin 120°}$$

which, multiplying both sides by $\sin\theta = \sin 30°$, gives $x = \frac{10\sin 30°}{\sin 120°} = \frac{10\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{10}{\sqrt{3}}$ or $\frac{10\sqrt{3}}{3}$.

   (b) **(5 points)** *Determine* x*:*



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3. **(12 points)** *Answer the following questions about trigonometric equations.*

   (a) **(4 points)** *Find* all *solutions to the equation* $6\sin(2x) = -3\sqrt{3}$.

Note below the procedure used to invert the sine, to ensure that *all* solutions are found:

$$\begin{aligned}
6\sin(2x) &= -3\sqrt{3} \\
\sin(2x) &= \frac{-\sqrt{3}}{2} \\
2x &= \arcsin\frac{-\sqrt{3}}{2} + 2\pi n \text{ or } \left(\pi - \arcsin\frac{-\sqrt{3}}{2}\right) + 2\pi n \\
2x &= \frac{-\pi}{3} + 2\pi n \text{ or } \frac{4\pi}{3} + 2\pi n \\
x &= \frac{-\pi}{6} + \pi n \text{ or } \frac{2\pi}{3} + \pi n
\end{aligned}$$

So x can take on any value of the form $\pi n - \frac{\pi}{6}$ or of the form $\pi n + \frac{2\pi}{3}$.

(b) **(4 points)** *Find* any one *solution to the equation* $4\sec(3x) - 2 = 6$.

Since we were only asked to find one solution, we don't have to be too cautious with our cosine-inversion, and can use the ordinary arc-cosine:

$$\begin{aligned}
4\sec(3x) - 2 &= 6 \\
\frac{4}{\cos(3x)} - 2 &= 6 \\
\frac{4}{\cos(3x)} &= 8 \\
4 &= 8\cos(3x) \\
\frac{1}{2} &= \cos(3x) \\
\arccos\frac{1}{2} &= 3x \\
\frac{\pi}{3} &= 3x \\
\frac{\pi}{9} &= x
\end{aligned}$$

There are several other acceptable answers, if a different choice of value whose cosine is $\frac{1}{2}$ is used.

(c) **(4 points)** *Verify the trigonometric identity* $\frac{1-\sin x}{1+\sin x} = (\sec x - \tan x)^2$.

We expand the right side:

$$\begin{aligned}(\sec x - \tan x)^2 &= \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)^2 \\ &= \left(\frac{1-\sin x}{\cos x}\right)^2 \\ &= \frac{(1-\sin x)^2}{\cos^2 x} \\ &= \frac{(1-\sin x)^2}{1-\sin^2 x} \\ &= \frac{(1-\sin x)^2}{(1-\sin x)(1+\sin x)} = \frac{1-\sin x}{1+\sin x}\end{aligned}$$

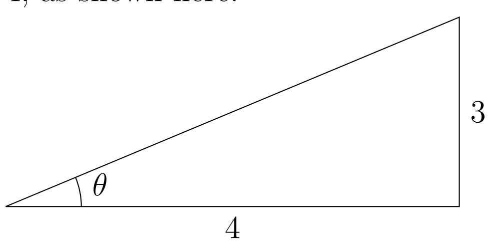
4. **(10 points)** *Answer the following questions about evaluating trigonometric expressions.*

(a) **(2 points)** *Evaluate* $\arctan(-1)$.

There are a few familiar tangents, and their associated arctangents should be discernable from the familiarity of their form. For instance, we know that $\tan \frac{-\pi}{4} = -1$, and thus $\arctan(-1) = \frac{-\pi}{4}$.

(b) **(4 points)** *Evaluate* $\csc\big(\arctan(\frac{3}{4})\big)$.

We give a simple name to $\arctan \frac{3}{4}$; traditionally we might call it θ. To convey this information we might build a triangle exemplifying this relationship between θ and $\frac{3}{4}$, which we might write as $\theta = \arctan \frac{3}{4}$, but more comprehensibly as $\tan\theta = \frac{3}{4}$; in a right triangle with θ as one of the angles, we know that $\tan\theta$ represents the ratio of the lengths of the opposite side and the adjacent side. We would thus represent this relationship by making the opposite side of the triangle have length 3, and the adjacent side have length 4, as shown here:



Furthermore, the hypotenuse, which is not labeled in the above picture, can be calculated by the Pythagorean Theorem to have length $\sqrt{3^2+4^2} = \sqrt{9-16} = \sqrt{25} = 5$.

After that setup, our actual calculation ends up being very easy! We wanted to find $\csc \arctan \frac{3}{4}$, which in light of our definition of θ can be written as simply $\csc\theta$. Finding any trigonometric identity of an angle in a labeled right triangle is simply a matter of dividing the appropriate sides: the cosecant is the ratio of the lengths of the hypotenuse and opposite side, so in this case, $\csc\theta = \frac{5}{3}$.

(c) **(2 points)** *Evaluate* $\arcsin \frac{1}{2}$.

There are a few familiar sines, and their associated arcsines should be discernable from the familiarity of their form. Recall that the arcsine function returns values between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$. We know that $\cos \frac{\pi}{6} = \frac{1}{2}$, and thus $\arcsin \frac{1}{2} = \frac{-\pi}{2}$.

(d) **(3 points)** *Evaluate the expression* $\cos(55°)\cos(10°) + \sin(55°)\sin(10°)$.

The above expression is a template which is recognizable as one of those covered in our angle-addition rules; recall that

$$\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$

whose right side matches nicely with the expression seen here, so

$$\cos(55°)\cos(10°) + \sin(55°)\sin(10°) = \cos(55° - 10°) = \cos 45° = \frac{\sqrt{2}}{2}$$

5. **(10 points)** *The following ten graphs are of the following functions:*

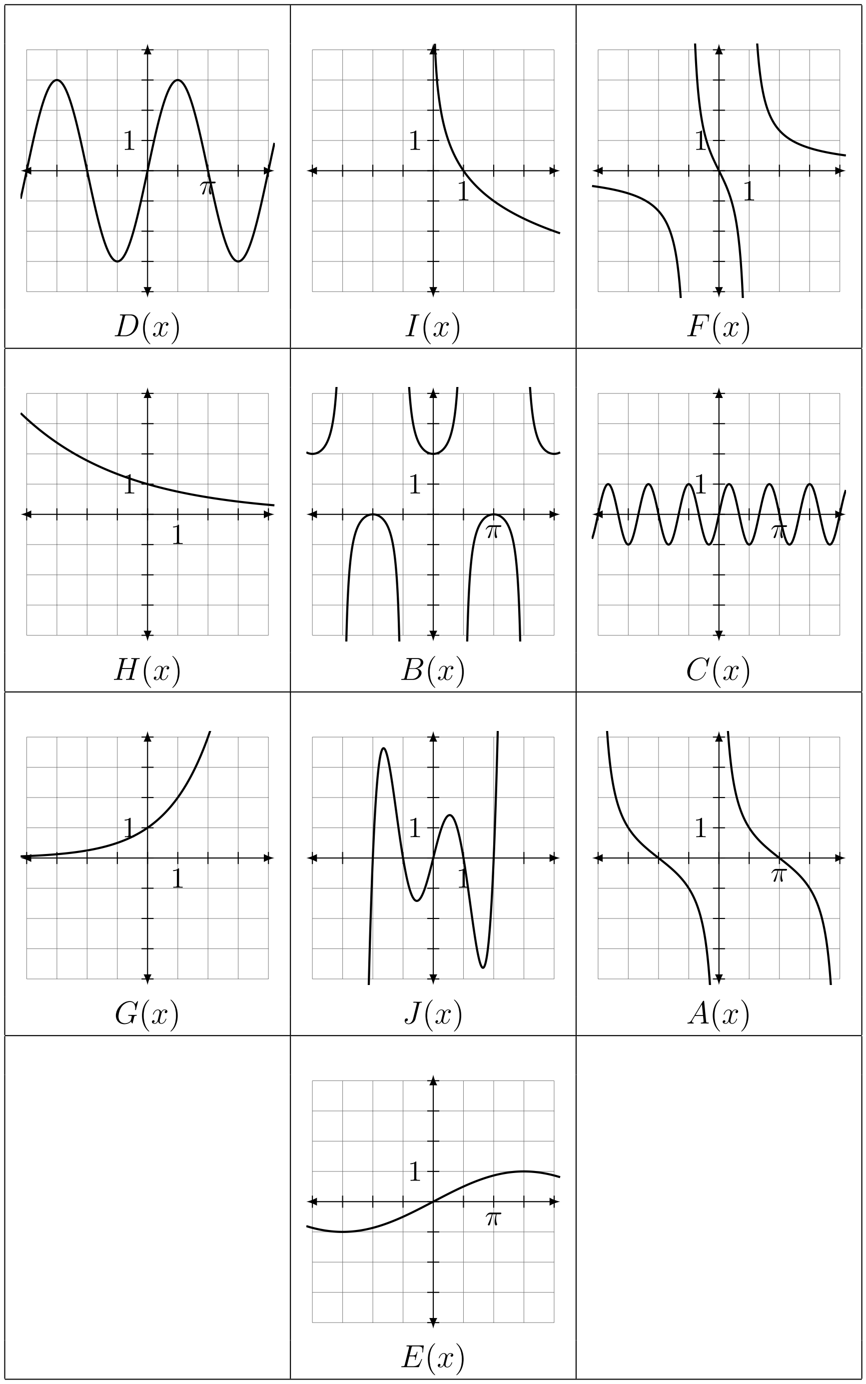
$$A(x) = \cot\left(\frac{x}{2}\right) \qquad B(x) = 1 + \sec x \qquad C(x) = \sin 3x$$

$$D(x) = 3\sin x \qquad E(x) = \sin\frac{x}{3} \qquad F(x) = \frac{1}{x-1} + \frac{1}{x+1}$$

$$G(x) = 2^x \qquad H(x) = \left(\frac{3}{4}\right)^x \qquad I(x) = \log_{1/2} x$$

$$J(x) = x(x+1)(x-1)(x+2)(x-2)$$

Label each picture with the letter of the appropriate function.



Explanations of the answers given above follow; descriptions are given for each picture row-by-row, working from left to right and top to bottom.

The first picture in the first row clearly shows a sine curve with a period of 2π and amplitude of 3; it thus matches the formula $D(x) = 3\sin x$.

The second picture in the first row clearly has a domain consisting only of positive numbers, since it ceases to exist on the left half of the plane. Both the shape and domain suggest a logarithm, and the only option in the list given is $I(x) = \log_{1/2} x$.

The third picture in the first row has discontinuities at $x = -1$ and $x = 1$ (in between it looks vaguely like a tangent curve, but it does not have the periodicity we would expect of a tangent or cotangent). We note that this finite list of discontinuities is consistent with a rational function where $x-1$ and $x+1$ appear in the denominatoris, which matches $F(x) = \frac{1}{x-1} + \frac{1}{x+1}$ perfectly.

The first picture in the second row is positive throughout and tending towards zero as x increases. This is the standard shape for a decay function, i.e. an exponential whose base is less than 1. The function $H(x) = \left(\frac{3}{4}\right)^x$ is the only such function in the list.

The second picture in the second row has a shape highly evocative of a secant or cosecant, and its phase suggests a secant in particular. One oddity is that instead of having local extrema at $y = 1$ and $y = -1$, this curve has the extrema at $y = 0$ and $y = 2$; we thus have a vertically shifted secant, as described by the function $B(x) = 1 + \sec x$.

The third picture in the second row is a short-period sine curve; noting that there are three periods squeezed into the interval $[0, 2\pi]$, we might characterize it as having period $\frac{2\pi}{3}$ and amplitude 1, which are the characteristics of the function $C(x) = \sin(3x)$.

The first picture in the third row is positive throughout and tending towards zero as x decreases. This is the standard shape for a growth function, i.e. an exponential whose base is greater than 1. The function $G(x) = 2^x$ is the only such function in the list.

The second picture in the third row has several zeroes at the values $x = -2$, $x = -1$, $x = 0$, $x = 1$, and $x = 2$. It does not appear to be periodic, so it is reasonable to conclude it is a rational or polynomial function; $J(x) = x(x+1)(x-1)(x+2)(x-2)$ has the specified zeroes, by construction.

The third picture in the third row has the characteristic shape of a tangent or cotangent, with orientation more like a cotangent. Its period is 2π instead of the mure usual π, so it is the horizontally stretched cotangent function $A(x) = \cot \frac{x}{2}$.

The picture in the fourth row is a sine curve with a very long period. We see *half* a period (form trough to crest) between $x = \frac{-3\pi}{2}$ and $x = \frac{3\pi}{2}$, so the period as a whole is 6π, and the amplitude is 1. These characteristics describe $E(x) = \sin \frac{x}{3}$.