1. **(10 points)** *Chrome Koran's latest album, made available for purchase online, would be bought by 4500 people at a price point of \$5. Fan polling suggests that every increase in the price by \$1 would reduce the number of downloads by 500.*

   (a) **(3 points)** *Find a function describing the* demand *for the album as a function of price.*

   Since demand scales directly with the price, demand is a linear function of price. If our demand function is $D(x) = mx + b$, then the fact that an increase of 1 in x corresponds with a decrease of 500 in $D(x)$ signifies that $m = -500$, so $D(x) = -500x + b$. Since $D(5) = 4500$, we may solve for b in the equation $-500 \cdot 5 + b = 4500$ to get $b = 7000$, so $D(x) = 7000 - 500x$.

   (b) **(3 points)** *Find a function describing the total* revenue *from album sales as a function of price.*

   At a price point x, the band will sell $D(x)$ albums, gaining revenue of x on each sale, for total revenue of $x \cdot D(x)$. Thus the revenue function is $R(x) = xD(x) = 7000x - 500x^2$.

   (c) **(4 points)** *Find a sale price for the album which maximizes revenue, and the total revenue earned at this price. Label which is which.*

   The function $R(x) = 7000x - 500x^2$ is a quadratic with negative quadratice coefficient; it is thus maximized at its vertex, whose x-coordinate is $\frac{-7000}{2 \cdot -500} = 7$ and whose y-coordinate is $R(x) = 7000 \cdot 7 - 500 \cdot 7^2 = 24500$, so they can earn a total revenue of \$24500 by selling their album at \$7 per copy.

2. **(14 points)** *Answer the following questions about the polynomial function $f(x) = 2x^3 + 5x^2 + x - 2$.*

   (a) **(3 points)** *What is the average rate of change of this function between the values $x = 0$ and $x = 2$?*

   The average rate of change is $\frac{f(2)-f(0)}{2-0} = \frac{36-(-2)}{2} = 19$.

   (b) **(3 points)** *What are all the* potential *rational zeroes of this function?*

   By the Rational Root Theorem, the possible zeroes are positive or negative fractions whose numerators are divisors of 2 and whose denominators are also divisors of 2. The possibilities are thus $\pm\frac{1}{2}$, ± 1, and ± 2.

   (c) **(4 points)** *Factor the polynomial into linear terms.*

   We may perform several synthetic divisions by the results in the previous question; one which is successful is the division by $(x + 2)$:

$$\begin{array}{r|rrrr} -2 & 2 & 5 & 1 & -2 \\ & & -4 & -2 & 2 \\ \hline & 2 & 1 & -1 & 0 \end{array}$$

   so $f(x) = (x + 2)(2x^2 + x - 1)$. The quadratic can be either factored or solved using the quadratic formula to find that $f(x) = (x + 2)(2x - 1)(x + 1)$.

   (d) **(4 points)** *What are the x-intercepts, y-intercept, and long term behavior of the function? Label which is which.*

   The x-intercepts ca be calculated from the previous question to be $x = -2$, $x = \frac{1}{2}$, and $x = -1$. The y-intercept is at $f(0) = -2$, and the long-term behavior, since this is a polynomial with leading term of odd degree and positive coefficient, is that as x

becomes large in the positive and negative directions, $f(x)$ will likewise become large in the positive and negative directions respectively.

3. **(6 points)** *Calculate the following trigonometric expressions.*

   (a) **(2 points)** $\arcsin \frac{\sqrt{3}}{2}$.
   Since $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{-\pi}{2} \leq \frac{\pi}{3} \leq \frac{\pi}{2}$, it follows that $\arcsin \frac{\sqrt{3}}{2} = \frac{\pi}{3}$.

   (b) **(2 points)** $\csc \frac{17\pi}{6}$.
   Since $\frac{17\pi}{6}$ describes a point with reference number $\frac{\pi}{6}$ in the second quadrant, its sine is $\frac{1}{2}$, so its cosecant is the reciprocal of $\frac{1}{2}$, which is 2.

   (c) **(2 points)** $\tan \frac{-5\pi}{3}$.
   Since $\frac{-5\pi}{3}$ describes a point with reference number $\frac{\pi}{3}$ in the first quadrant, its tangent is $\sqrt{3}$.

4. **(10 points)** *Answer the following questions about logarithms.*

   (a) **(3 points)** *Find a value of* x *such that* $3 \cdot 2^{2x-1} + 5 = 53$.
   We can algebraically isolate x step-by-step:

$$\begin{aligned} 3 \cdot 2^{2x-1} + 5 &= 53 \\ 3 \cdot 2^{2x-1} &= 48 \\ 2^{2x-1} &= 16 \\ 2x - 1 &= \log_2 16 = 4 \\ 2x &= 5 \\ x &= \tfrac{5}{2} \end{aligned}$$

   (b) **(3 points)** *Calculate the value of the expression* $\log_5 140 + \log_5 \frac{2}{7} - 3\log_5 10$ *exactly.*
   Using known logarithm laws:

$$\log_5 140 + \log_5 \frac{2}{7} - 3\log_5 10 = \log_5 140 + \log_5 \frac{2}{7} - \log_5 (10)^3 = \log_5 \frac{140 \cdot 2}{7 \cdot 1000} = \log_5 125 = -2.$$

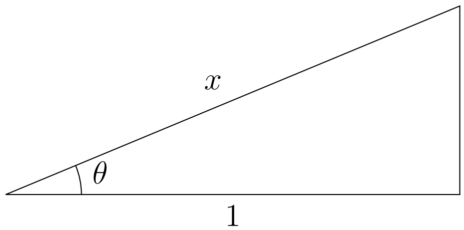
   (c) **(4 points)** *Calculate the folowing logarithms exactly, giving numerical answers:*

   - $\log_4 8$.
     Since $8 = (\sqrt{4})^3$, we know that $8 = 4^{2/3}$. Thus $\log_4 8 = \frac{2}{3}$.
   - $\log_7 49$.
     Since $49 = 7^2$, $\log_7 49 = 2$.
   - $\log_3 \frac{1}{27}$.
     $\frac{1}{27} = \frac{1}{3^3} = 3^{-3}$, and so $\log_3 \frac{1}{27} = -3$.
   - $\log_5 5$.
     $5 = 5^1$, so $\log_5 5 = 1$.

5. **(10 points)** *Answer the following trigonometric questions.*

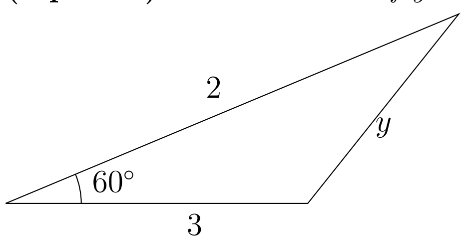
(a) **(4 points)** *Simplify the expression* $\tan(\operatorname{arcsec} x)$ *to a form which does not use trigonometric functions.*

If we define θ to equal $\operatorname{arcsec} x$, then we could equivalently assert that $x = \sec\theta$. We could then depict that relationship as between the angles and sides of a rigt triangle: since the secant of an angle is the ratio of the hypotenuse of a right triangle to its adjacent side, we could then label two edges of the right triangle in the ratio $\frac{x}{1}$:



The unlabeled leg may be found by the Pythagorean theorem to have side length $\sqrt{x^2-1}$. Then, since the tangent is the ratio of the lengths of the opposite and adjacent sides, it follows that $\tan\theta = \frac{\sqrt{x^2-1}}{1} = \sqrt{x^2-1}$.

(b) **(3 points)** *Find the value of* y *in the triangle (not drawn to scale) below.*



Since the labeled elements of this triangle are three sides and an angle, we can use the Law of Cosines to relate them, as such:

$$y^2 = 2^2 + 3^3 - 2 \cdot 2 \cdot 3 \cdot \cos 60^\circ$$

which solves to give $y^2 = 7$, so $y = \sqrt{7}$ (we may ignore the negative solution, as it does not make sense for a length to be negative).

(c) **(3 points)** *If* θ *describes a point in quadrant III and* $\cos\theta = \frac{-2}{3}$, *what is* $\sin(2\theta)$*?*

We know from a double-angle formula that $\sin 2\theta = 2\sin\theta\cos\theta$, but we do not actually know $\sin\theta$! We can find it using our knowledge of $\cos\theta$, though:

$$\begin{aligned}
\sin^2\theta + \cos^2\theta &= 1 \\
\sin^2\theta + \left(\tfrac{-2}{3}\right)^3 &= 1 \\
\sin^2\theta + \tfrac{4}{9} &= 1 \\
\sin^2\theta &= \tfrac{5}{9} \\
\sin\theta &= \pm\tfrac{\sqrt{5}}{3}
\end{aligned}$$

Note that since θ is in quadrant III, its sine is negative, so $\sin\theta = \frac{-\sqrt{5}}{3}$. Then, $\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{-\sqrt{5}}{3}\right)\left(\frac{-2}{3}\right) = \frac{4\sqrt{5}}{9}$.

6. **(15 points)** *Answer the following questions about the functions* $f(x) = \frac{\log_2(x+1)}{3x-2}$ *and* $g(x) = \sqrt{x-1}$. *In each question asking for multiple answers,* label which is which.

(a) **(3 points)** *Write formulas, which need not be simplified, for $f(g(x))$ and $g(g(x))$.*

$f(g(x)) = \frac{\log_2(\sqrt{x-1}+1)}{3\sqrt{x-1}-2}$ and $g(g(x)) = sqrt\sqrt{x-1}-1$.

(b) **(3 points)** *Determine the domains of $f(x)$ and $g(x)$.*

$f(x)$ has several potential causes of unevaluatability: there is a logarithm, whose parameter must be positive, and a division, whose denominator must be nonzero. Thus $x+1>0$ and $3x-2 \neq 0$, which simplifies to the requirements that $x > -1$ and $x \neq \frac{2}{3}$. In interval form, this would be $(-1, \frac{2}{3}) \cup (\frac{2}{3}, +\infty)$.

$g(x)$, on the other hand, will only be problematic when the parameter of its square root is negative, so we mandate that $x-1 \geq 0$, so $x \geq 1$, or in interval form, the domain is $[1, +\infty)$.

(c) **(2 points)** *Write formulas, which need not be simplified, for $(fg)(x)$ and $(f+g)(x)$.*

$(fg)(x) = \frac{\log_2(x+1)}{3x-2}\sqrt{x-1}$, and $(f+g)(x) = \frac{\log_2(x+1)}{3x-2} + \sqrt{x-1}$

(d) **(4 points)** *Determine the domains of $(f+g)(x)$ and $\frac{f}{g}(x)$.*

Both functions are evaluatable when both $f(x)$ and $g(x)$ are evaluatable, which is on $[1, +\infty)$.

(e) **(3 points)** *Find the inverse of the function $g(x)$.*

7. **(13 points)** *Answer the following questions about growth and decay.*

(a) **(2 points)** *An investment in the Kék Rózsa fund will increase in value by 3% each year. What will the value of a \$1000 initial investment be in 2 years?*

It will increase in the first year to \$1000(1.03) = \$1030; in the second it will increase further to \$1030(1.03) = \$1060.90.

(b) **(2 points)** *Solarbonite has a radioactive half-life of 50 hours. Produce a function describing the quantity of solarbonite remaining in a 20-gram sample after t hours.*

For the given initial quantity and half-life, a function for the amount remaining after t hours is $f(t) = 20\left(\frac{1}{2}\right)^{t/50}$.

(c) **(3 points)** *Unshielded human exposure to quantities of solarbonite in excess of 8 grams is generally regarded as unsafe. How long will one need to wait until a 20-gram sample of solarbonite reaches a safe level?*

We with to determine when $f(t) = 8$ above, and we may find it by algebraic isolation of t:

$$\begin{aligned} 20\left(\tfrac{1}{2}\right)^{t/50} &= 8 \\ \left(\tfrac{1}{2}\right)^{t/50} &= \tfrac{8}{20} = \tfrac{2}{5} \\ \tfrac{t}{50} &= \log_{1/2}\tfrac{2}{5} \\ t &= 50\log_{1/2}\tfrac{3}{5} = \frac{50\ln\frac{2}{5}}{\ln\frac{1}{2}} \end{aligned}$$

This quantity is approximately 66.1 hours, although you could not reasonably find that approximation without a calculator.

(d) **(3 points)** *The population of Santa Carla is currently 12000. Over the course of each year the population will grow by 2%. How many years will it take for the population to reach 20000?*

The population after t years will be $12000(1.02)^t$, so we solve for when that equals 20000:

$$\begin{aligned} 12000(1.02)^t &= 20000 \\ 1.02^t &= \tfrac{20000}{12000} = \tfrac{5}{3} \\ t &= \log_{1.02} \tfrac{5}{3} = \frac{\ln \frac{5}{3}}{\ln 1.02} \end{aligned}$$

This quantity is approximately 25.8 years, but that could not be found without a calculator.

(e) **(3 points)** *A frozen pizza is taken out of a freezer and placed into a hot oven; its temperature in degrees Fahrenheit t minutes after being moved is $f(t) = 400 - 380e^{-0.03t}$. What is the original temperature of the pizza and the temperature of the oven, and how long will it take to heat to $200°F$? Label each of your answers.*

The initial temperature is $f(0) = 400 - 380e^0 = 20°$. To find when it reaches $200°$, we must solve the equation $f(t) = 200$:

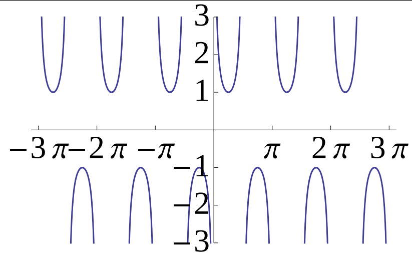
$$\begin{aligned} 400 - 380e^{-0.03t} &= 200 \\ -380e^{-0.03t} &= -200 \\ e^{-0.03t} &= \tfrac{-200}{-380} = \tfrac{10}{19} \\ -0.03t &= \ln \tfrac{10}{19} \\ t &= \frac{\ln \frac{10}{19}}{-0.03} \end{aligned}$$

which is actually about 21.4 minutes.

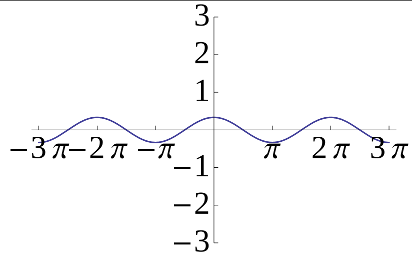
8. **(12 points)** *The following twelve graphs are of the following functions:*

$$\begin{array}{llll} A(x) = \ln x & B(x) = 2^x & C(x) = \dfrac{x}{(x-1)(x+2)} & D(x) = \dfrac{(x-1)(x+2)}{x} \\ E(x) = 3\cos x & F(x) = \cos\dfrac{x}{3} & G(x) = \cos(3x) & H(x) = \dfrac{1}{3}\cos x \\ I(x) = 1 + \sin x & J(x) = \left(\dfrac{1}{3}\right)^x & K(x) = \cot x & L(x) = \csc 2x \end{array}$$

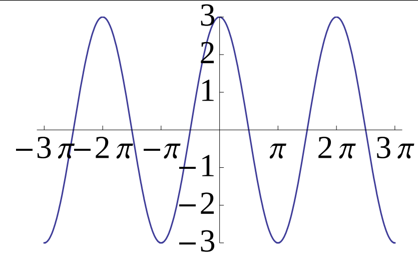
Label each picture with the letter of the appropriate function.



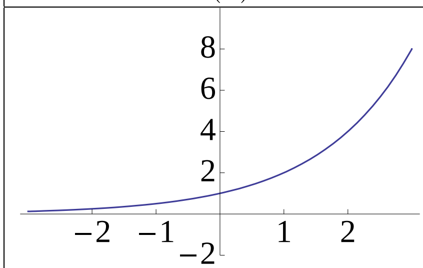
$L(x)$



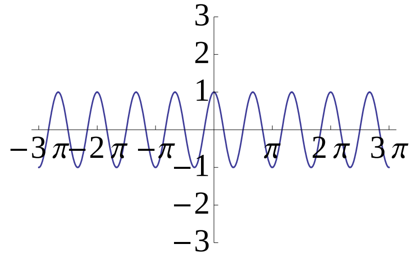
$H(x)$



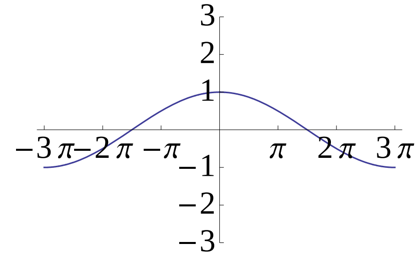
$E(x)$



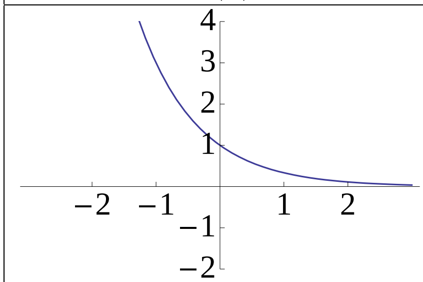
$B(x)$



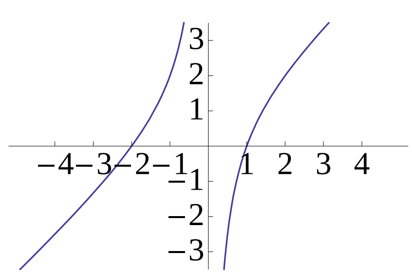
$G(x)$



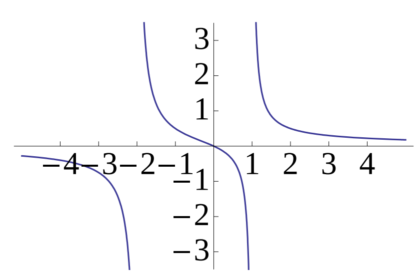
$F(x)$



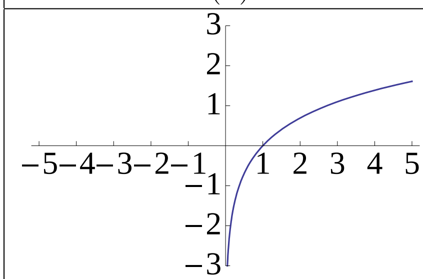
$J(x)$



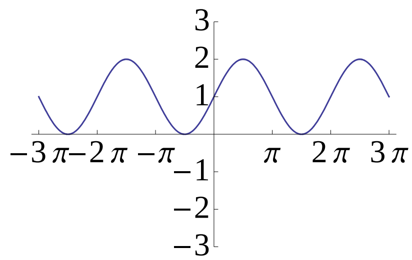
$D(x)$



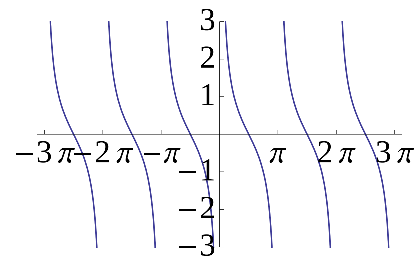
$C(x)$



$A(x)$



$I(x)$



$K(x)$