1. **(9 points)** *Answer the following questions about the quadratic function* $f(x) = -3x^2 + 12x - 4$.

   (a) **(1 point)** *Calculate the y-intercept of this function.*

   Since $f(0) = -4$, the y-intercept is -4.

   (b) **(3 point)** *Calculate the x-intercepts, if any, of this function. If it has no x-intercepts, explicitly say so.*

   To determine where $f(x) = 0$, we solve the quadratic equation $-3x^2 + 12x - 4 = 0$ to get the roots

   $$x = \frac{-12 \pm \sqrt{12^2 - 4(-3)(-4)}}{2(-3)} = \frac{-12 \pm \sqrt{96}}{-6} = 2 \pm \frac{2\sqrt{6}}{3}$$

   (c) **(3 points)** *Write the above quadratic function in standard form.*

   We may complete the square:

   $$\begin{aligned} f(x) &= -3x^2 + 12x - 4 \\ &= -3(x^2 - 4x) - 4 \\ &= -3(x^2 - 4x + 4 - 4) - 4 \\ &= -3(x-2)^2 + 8 \end{aligned}$$

   (d) **(2 points)** *What are the coordinates of the vertex of this function's graph? Is the vertex a maximum, a minimum, or neither?*

   From the above, we can find that the vertex (h, k) is $(2, 8)$ (alternatively we can compute $h = \frac{-b}{2a} = \frac{-12}{2(-3)} = 2$ and $k = f(h) = -3 \cdot 2^2 + 12 \cdot 2 - 4 = 8$). This is a maximum because the quadratic coefficient is negative so the graph is concave downwards.

2. **(11 points)** *Let* $f(x) = 3x^3 - 5x^2 - 8x - 2$. *Answer the following questions about this polynomial function.*

   (a) **(1 point)** *Calculate the y-intercept of this function.*

   Since $f(0) = -2$, the y-intercept is -2.

   (b) **(3 points)** *Briefly describe the long-term behavior of this function on both ends, i.e., describe the value of* $f(x)$ *when* x *is a very large positive number, and when* x *is a large-magnitude negative number.*

   The leading term is $3x^3$, which has a positive coefficient and an odd exponent, so it is large and positive when x is large and positive, and extremely negative when x is extremely negative, like the function $g(x) = x^3$ is.

   (c) **(3 points)** *Determine, without actually performing any divisions, which values* could *be rational roots of the polynomial.*

   Using the rational root theorem, the numerator of any rational root must be a positive or negative factor of 2, i.e., either ± 1 or ± 2, while the denominator must be a factor of 3, i.e. 1 or 3, for a complete list of

   $$1, -1, 2, -2, \frac{1}{3}, \frac{-1}{3}, \frac{2}{3}, \frac{-2}{3}.$$

(d) **(4 points)** *Calculate all the real roots of the polynomial.*

We perform trial synthetic or long division with the eight potential rational roots; for your edification the results of all of these synthetic divisions are included below,a lthough only one rational root is actually required. Note that several processes could be terminated early as soon as a fraction appears.

$$\begin{array}{r|rrrr} 1 & 3 & -5 & -8 & -2 \\ & & 3 & -2 & -10 \\ \hline & 3 & -2 & -10 & -12 \end{array} \qquad \begin{array}{r|rrrr} -1 & 3 & -5 & -8 & -2 \\ & & -3 & 8 & 0 \\ \hline & 3 & -8 & 0 & -2 \end{array}$$

$$\begin{array}{r|rrrr} 2 & 3 & -5 & -8 & -2 \\ & & 6 & 2 & -12 \\ \hline & 3 & 1 & -6 & -14 \end{array} \qquad \begin{array}{r|rrrr} -2 & 3 & -5 & -8 & -2 \\ & & -6 & 22 & -28 \\ \hline & 3 & -11 & 14 & -30 \end{array}$$

$$\begin{array}{r|rrrr} \frac{1}{3} & 3 & -5 & -8 & -2 \\ & & 1 & \frac{-4}{3} & \frac{-28}{9} \\ \hline & 3 & -4 & \frac{-28}{3} & \frac{-46}{9} \end{array} \qquad \begin{array}{r|rrrr} \frac{-1}{3} & 3 & -5 & -8 & -2 \\ & & -1 & 2 & 2 \\ \hline & 3 & -6 & -6 & 0 \end{array}$$

$$\begin{array}{r|rrrr} \frac{4}{3} & 3 & -5 & -8 & -2 \\ & & 4 & \frac{-4}{3} & \frac{-112}{9} \\ \hline & 3 & -1 & \frac{-28}{3} & \frac{-130}{9} \end{array} \qquad \begin{array}{r|rrrr} \frac{-4}{3} & 3 & -5 & -8 & -2 \\ & & -4 & 12 & \frac{-16}{3} \\ \hline & 3 & 4 & -6 & \frac{-22}{3} \end{array}$$

Thus, a successful trial synthetic or long division by the potential factor $x - \frac{-1}{3}$ reveals that $f(x) = \left(x + \frac{1}{3}\right)(3x^2 - 6x - 6)$. One root is clearly $\frac{-1}{3}$; the other two are roots of $3x^2 - 6x - 6$ and can thus be obtained by the quadratic formula to be $x = \frac{6 \pm \sqrt{36+72}}{6} = 1 \pm \sqrt{3}$. Thus, the roots of the original cubic are $\frac{-1}{3}$, $1 + \sqrt{3}$, and $1 - \sqrt{3}$.