1. **(20 points)** For each of the following relations R on given sets S, determine whether each of the reflexive, symmetric, and transitive properties hold. Briefly justify your claims.
   (a) $S = \{1, 2, 3\}$, $R = \{(1, 2), (1, 3), (2, 2), (2, 3), (3, 2), (3, 3)\}$.
   (b) $S = \mathcal{P}(\mathbb{N})$, with R given by the criterion that $S\ R\ T$ if and only if $S \cap T = \emptyset$.
   (c) $S = \mathbb{R}$, with R given by the criterion that $x\ R\ y$ if and only if $|x - y| \leq 1$.

2. **(18 points)** Formally prove that $A - (B \cap C) = (A - B) \cup (A - C)$.

3. **(18 points)** Identify each of the following statements as a tautology, a contradiction, or neither. Show your work.
   (a) $(P \vee Q) \Rightarrow (P \wedge Q)$.
   (b) $([\sim P] \Rightarrow Q) \vee (Q \Leftrightarrow P)$.
   (c) $[(P \Rightarrow Q) \Rightarrow P] \Rightarrow P$.

4. **(20 points)** Identify each of the following functions with the stated domains and codomains, as injective, surjective, bijective, or none of the above. Briefly justify your claim.
   (a) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = 2n + 1$.
   (b) $g : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ given by $g(S) = S \cup \{1, 2, 3, 4, 5\}$.
   (c) $h : \mathbb{N} \to \mathbb{N} \cup \{0\}$ given by $h(2^k m) = k$ whenever m is odd; that is to say, $h(n)$ is equal to the exponent of 2 present in the prime factorization of n, so $h(40) = 3$ and $h(37) = 0$.

5. **(24 points)** Let R_1 and R_2 be relations on a set S. Below, set operations are performed on the relations considered as subsets of $S \times S$.
   (a) Prove that $R_1 \cap R_2$ is reflexive *if and only if* both R_1 and R_2 are reflexive.
   (b) Prove that if both R_1 and R_2 are symmetric, then the relation $R_1 \cap R_2$ is also symmetric.
   (c) Prove that if both R_1 and R_2 are transitive, then the relation $R_1 \cap R_2$ is also transitive.

6. **(18 points)** Prove that for any natural number n,
$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}.$$

7. **(20 points)** Let $S = \big\{\emptyset, 2, 3, \{5, 7, 9\}, \{1, 2, 3, 4, \ldots\}\big\}$. For each of the following descriptions, either produce a set matching the description or explain briefly why such a set doesn't exist.
   (a) A set A of 4 elements such that $A \in S$.
   (b) A set B of 4 elements such that $B \subseteq S$.
   (c) A set C of 4 elements, such that $C \in \mathcal{P}(S)$.
   (d) A set D of 4 elements, such that $D \subseteq \mathcal{P}(S)$.

8. **(12 points)** Prove or disprove: there is a natural number n such that $5 \mid (3^n - 1)$.

The best of ideas is hurt by uncritical acceptance and thrives on critical examination.
—George Polya, *How to Solve It*