1. **(32 points)** *Consider the linear programming problem in which we maximize the profit function* $3x + 10y$ *subject to the conditions*

$$\begin{cases} 3x + 2y \leq 90 \\ x + 3y \leq 60 \\ y \leq 15 \\ x \geq 0, y \geq 0 \end{cases}$$

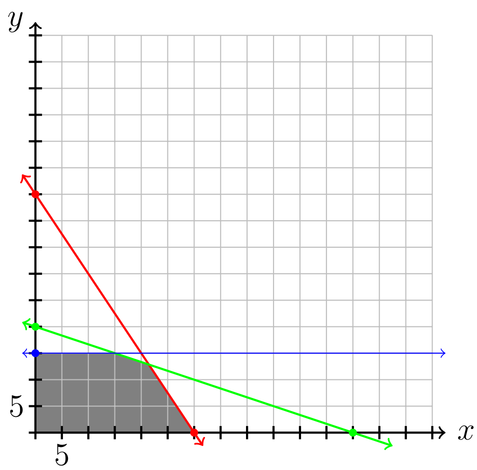
(a) **(10 points)** *Sketch the constraint lines, labeling the intercepts, on the axes below, and shade the feasible region (indicate the scale on the axes).*

The line $3x + 2y = 90$ has an x-intercept of $\frac{90}{3} = 30$ and a y-intercept of $\frac{90}{2} = 45$.

The line $x + 3y = 60$ has an x-intercept of $\frac{60}{1} = 60$ and a y-intercept of $\frac{60}{3} = 20$.

The line $y = 15$, since it has no x-component, is a horizontal line with the y-intercept of 15.

We can draw all of these on the axes by finding the intercepts and connecting them up with appropriate lines; we then shade the region under and to the left of these lines which is above and to the right of the axes:



(b) **(14 points)** *Find the coordinates for each potential feasible profit-maximizing point in the graph above.*

There are five corners to the region depicted in the graph, three of which are easily identified: the origin (0,0), the x-intercept (39,0), and the y-intercept (0,15). The other two points are the intersections of certain constraint lines: namely, the intersection of $x+3y = 60$ with each of $y = 15$ and $3x + 2y = 90$.

For the first of these, we want to solve the simultaneous system of equations

$$\begin{cases} x + 3y = 60 \\ y = 15 \end{cases}$$

We can plug our known value $y = 15$ into the first equation to solve for x:

$$x + 3 \cdot 15 = 60$$
$$x = 60 - 3 \cdot 15 = 15$$

so this point has coordinates (15,15).

The other point is at the intersection of the lines $x + 3y = 60$ and $3x + 2y = 90$; we thus want to solve the simultaneous system of equations

$$\begin{cases} x + 3y = 60 \\ 3x + 2y = 90 \end{cases}$$

which we can do by multiplying the former equation by 3 and subtracting the latter, leaving $7y = 90$, so $y = \frac{90}{7} \approx 12.9$. To find y, we plug our known value of y back in to either equation and solve for y:

$$\begin{aligned} x + 3 \cdot \frac{90}{7} &= 60 \\ x &= 60 - 3 \cdot \frac{90}{7} = \frac{150}{7} \\ x &= \frac{150}{7} \approx 21.4 \end{aligned}$$

so our coordinates for this point are about (21.4,12.9).

Our list of possible maximizing points is thus $\boxed{\text{(0,0), (0,15), (15,15), (21.4,12.9), and (0,30).}}$

If you like, you may drop (or not even consider in the first place) either or both of the points (0,0) or (0,15) as clearly inferior, leaving the shortened list $\boxed{\text{(15,15), (12.9,21.4), and (30,0).}}$

(c) **(8 points)** *Find the value of the pair (x,y) maximizing the profit on the above graph.*

With the three potential profit maximizers determined in the last part, we test each one to determine the actual profit:

$$\begin{aligned} \text{At (15.0,15.0)} &: 3 \times 15.0 + 10 \times 15.0 = 195 \\ \text{At (21.4,12.9)} &: 4 \times 21.4 + 10 \times 12.9 \approx 214.3 \\ \text{At (30,0 \ 0.0)} &: 3 \times 30.0 + 10 \times \ \ 0.0 = \ \ 90 \end{aligned}$$

Thus, the choice $\boxed{(21.4,12.9)}$ maximizes the profit.

2. **(32 points)** *You are producing children's bicycles, tricycles, and wagons in a workshop. Each bicycle uses two wheels and 3 feet of steel tubing, and can be sold for a profit of \$25. Tricycles each require three wheels, 4 feet of steel tubing, and a square foot of sheet metal; each tricycle yields \$40 in profit. Finally, wagons each require four wheels, use 2 feet of steel tubing each, and 5 square feet of sheet metal; these can be sold for a profit of \$35. You have 400 wheels, 500 feet of steel tubing, and 300 square feet of sheet metal.*

(a) **(14 points)** *Produce a linear-programming formulation of this scenario, including constraints and a goal.*

We shall associate names with the production variables: then we might call x the number of bicycles made, y the number of tricycles, and z the number of wagons. Then the profit will be $25x + 40y + 35z$, and we will have three resource constraints defined by our limited quantities of wheels, steel tubing, and sheet metal.

Since each bicycle has 2 wheels, each tricycle 3, and each wagon 4, we use a total of $2x+3y+4z$ wheels in our selected production; since we only have 400 wheels in total, it is thus the case that $2x + 3y + 4z \leq 400$. Likewise, we will use $3x + 4y + 2z$ feet of steel tubing, which measured up against supplies of tubing gives the constraint $3x + 4y + 2z \leq 500$. Finally, we need $y + 5z$ square feet of sheet metal, so $y + 5z \leq 300$. Together with our limitations that production must be non-negative, the linear programming formulation of this problem is:

$$\boxed{\begin{array}{l}\text{Maximize } 25x + 40y + 35z \text{ subject to}\\ \left\{\begin{aligned} 2x + 3y + 4z &\leq 400\\ 3x + 4y + 2z &\leq 500\\ y + 5z &\leq 300\\ x &\geq 0\\ y &\geq 0\\ z &\geq 0 \end{aligned}\right.\end{array}}$$

(b) *Could you produce 60 bicycles, 50 tricycles, and 40 wagons with the available materials? If not, why not, and if so, how much profit would you make and how much slack would remain in each constraint?*

We test the truth of the resource constraints for the specific values $x = 60$, $y = 50$, and $z = 40$:

$$\begin{aligned} 2 \times 60 + 3 \times 50 + 4 \times 40 &= 430 \not\leq 400\\ 3 \times 60 + 4 \times 50 + 2 \times 40 &= 460 \leq 500\\ 50 + 5 \times 40 &= 250 \leq 300 \end{aligned}$$

Since the first resource constraint fails, this plan is $\boxed{\text{infeasible.}}$ If you wanted to be specific about the failure modes in the real world you could specifically say there are not enough wheels (which is what the first constraint corresponds to).

(c) *Could you produce 40 bicycles, 40 tricycles, and 50 wagons with the available materials? If not, why not, and if so, how much profit would you make and how much slack would remain in each constraint?*

We test the truth of the resource constraints for the specific values $x = 40$, $y = 40$, and $z = 50$:

$$\begin{aligned} 2 \times 40 + 3 \times 40 + 4 \times 50 &= 400 \leq 400\\ 3 \times 40 + 4 \times 40 + 2 \times 50 &= 380 \leq 500\\ 40 + 5 \times 50 &= 290 \leq 300 \end{aligned}$$

Since all resource constraints are met, this plan is $\boxed{\text{feasible,}}$ specifically producing a profit of

$$\$25 \times 40 + \$40 \times 40 + \$35 \times 50 = \boxed{\$4{,}350.}$$

The wheel constraint is in fact $\boxed{\text{tight,}}$ or could alternatively be described as having a slack of $\boxed{0.}$ A great deal of tubing is left over, leaving a slack of $500 - 380 = \boxed{120}$ in this constraint. Finally, the sheet-metal constraint has a slack of $300 - 290 = \boxed{10.}$

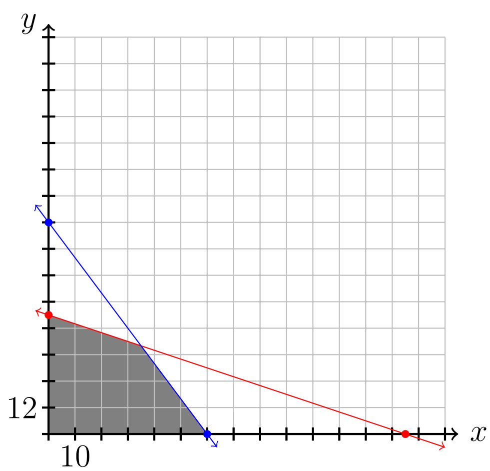
3. **(36 points)** *Joanna has borrowed a 3D printer and will be using it to make fractal-shaped crafts to sell at a makerspace festival. She can program the printer to make a pair of Menger-sponge earrings with 4 grams of filament and 8 minutes of construction time; she can also make a*

Sierpinski-triangle trivet in 5 minutes using 10 grams of filament. Her earrings will sell for a profit of \$5 and the trivets for a profit of \$8. If she only has a single 540-gram spool of filament and has the printer for only 480 minutes (8 hours), then how many of each product should she make, and how much profit will that production plan create?

This problem needs to be solved in stages. First we attach the production variables x and y to the numbers of earrings and trivets made respectively. There are two resource constraints: plastic filament, of which she has 540 grams, and time, of which she has 480 minutes. Thus her filament weight $4x + 10y$ cannot exceed 540, and her total production time $8x + 5y$ cannot exceed 480. Finally, her profit function is $5x + 8y$, so the mathematical formulation is to maximize $5x + 8y$ subject to:

$$\begin{cases} 4x + 10y \leq 540 \\ 8x + 5y \leq 480 \\ \qquad\quad x \geq 0 \\ \qquad\quad y \geq 0 \end{cases}$$

The first constraint has intercepts (135,0) and (0,54), while the second has intercepts (60,0) and (0,96). Plotting these on a graph gives us a feasible region.



Three of the corners of the region are obvious: (0,0), (60,0), and (0,54). However, the fourth is the intersection of the two constraint lines:

$$\begin{cases} 4x + 10y = 540 \\ 8x + 5y = 480 \end{cases}$$

By quadrupling the second equation and subtracting the first, we get $12x = 420$, so $x = \frac{420}{12} = 35$. Then $5y = 480 - 8x = 200$, so $y = 40$, giving (35,40) as the last corner point.

Finally, we test all four corners to see which maximizes the profit function:

$$\begin{aligned} \text{At } (\ 0,\ 0) &: 5 \times 0 + 8 \times 0 = 0 \\ \text{At } (60,\ 0) &: 5 \times 60 + 8 \times 0 = 300 \\ \text{At } (35,40) &: 5 \times 35 + 8 \times 40 = 495 \\ \text{At } (\ 0,54) &: 5 \times 0 + 8 \times 54 = 432 \end{aligned}$$

so her best strategy is to make *35 earrings and 40 trivets* to realize a profit of $\boxed{\$495.}$