1. **(30 points)** *The Loco Cocoa company produces tins of three different blends of hot chocolate: the light and easygoing Fireside blend, the rich and bitter Midnight blend, and the complex and spicy Mayan blend. A tin of Fireside uses six ounces of sugar, six ounces of powdered milk, and three ounces of cocoa powder, and can be sold for a profit of $5. A tin of Midnight uses four ounces of sugar, three ounces of milk, and five ounces of cocoa, and can be sold for a profit of $7. Lastly, a tin of Mayan uses four ounces of sugar, five ounces of milk, four ounces of cocoa, and a sixteenth of an ounce (a pinch!) of cayenne pepper, and can be sold for a profit of $8.*

   The company's latest shipment of supplies has 4320 ounces of sugar, 4800 ounces of powdered milk, 3360 ounces of cocoa powder, and 40 ounces of cayenne.

   (a) **(14 points)** *Produce a linear-programming formulation of this scenario, including constraints and a profit function.*

   We shall associate names with the production variables: then we might call x the number of tins of Fireside made, y the number of jugs of tins of Midnight, and z the number of tins of Mayan. Then the profit will be $5x + 7y + 8z$, and we will have four resource constraints defined by our limited quantities of sugar, powdered milk, cocoa, and cayenne powder.

   Since each tin of Fireside uses 6 ounces of sugar, each tin of Midnight uses four, and each tin of Mayan uses four, they use a total of $6x+4y+4z$ ounces of sugar in our selected production; since they have only 4320 ounces of sugar in total, we know that $6x + 4y + 4z \leq 4320$. Likewise, they use $6x + 3y + 5z$ ounces of milk, which measured up against supplies of milk gives the constraint $6x+3y+5z \leq 4800$. They also use $3x+5y+4z$ ounces of cocoa powder to make their products, so $3x + 5y + 4z \leq 3360$; and finally, $\frac{1}{16}z$ ounces of cayenne pepper are used so $\frac{1}{16}z \leq 40$. Together with our limitations that production must be non-negative, the linear programming formulation of this problem is:

$$\text{Maximize } 5x + 7y + 8z \text{ subject to } \begin{cases} 6x + 4y + 4z \leq 4320 \\ 6x + 3y + 5z \leq 4800 \\ 3x + 5y + 4z \leq 3360 \\ \frac{1}{16}z \leq 40 \\ x \geq 0 \\ y \geq 0 \\ z \geq 0 \end{cases}$$

   (b) **(9 points)** *Could they produce 600 tins of Fireside, 100 of Midnight, and 100 of Mayan with the materials they have on hand? If not, why not, and if so, how much profit would they make?*

   We test the truth of the resource constraints for the specific values $x = 600$, $y = 100$, and $z = 100$:

$$\begin{aligned} 6 \times 600 + 4 \times 100 + 4 \times 100 &= 4400 \quad \not\leq 4320 \\ 6 \times 600 + 3 \times 100 + 5 \times 100 &= 4400 \quad \leq 4800 \\ 3 \times 600 + 5 \times 100 + 4 \times 100 &= 2700 \quad \leq 3360 \\ \frac{1}{16} \times 100 &= 6.25 \leq 40 \end{aligned}$$

Since the first resource constraint fails, this plan is $\boxed{\text{infeasible.}}$ If you wanted to be specific about the failure modes in the real world you could specifically say there is not enough sugar (which is what the first constraint corresponds to).

(c) **(9 points)** *Could they produce 300 tins of Fireside, 200 of Midnight, and 300 of Mayan with the materials they have on hand? If not, why not, and if so, how much profit would they make?*

We test the truth of the resource constraints for the specific values $x = 300$, $y = 200$, and $z = 300$:

$$\begin{aligned} 6 \times 300 + 4 \times 200 + 4 \times 300 &= 3800 &\leq 4320 \\ 6 \times 300 + 3 \times 200 + 5 \times 300 &= 3900 &\leq 4800 \\ 3 \times 300 + 5 \times 200 + 4 \times 300 &= 3100 &\leq 3360 \\ \frac{1}{16} \times 300 &= 18.75 &\leq 40 \end{aligned}$$

Since all resource constraints are met, this plan is $\boxed{\text{feasible,}}$ specifically producing a profit of

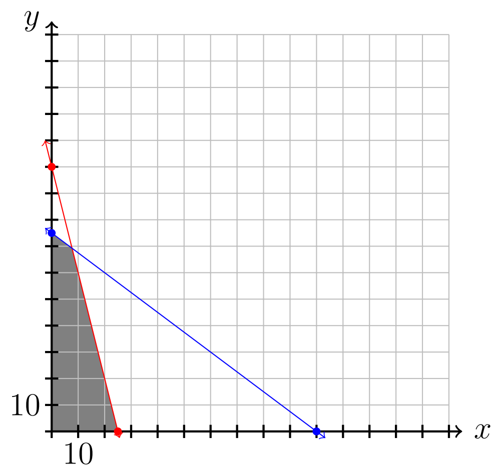
$$\$5 \times 300 + \$7 \times 200 + \$8 \times 300 = \boxed{\$5{,}300.}$$

2. **(36 points)** *You are making necklaces and earrings from woven wire and gemstones. Each necklace requires 4 feet of wire and 3 gemstones, while each pair of earrings requires one foot of wire and 4 gemstones. Necklaces sell for a profit of $35 each and pairs of earrings for $10 each. Each week you get 100 feet of wire and 300 gemstones. What combination of products can you produce each week to maximize your profit? Decimal answers are allowed; they'd represent a weekly average production.*

This problem needs to be solved in stages. First we attach the production variables x and y to the numbers of necklaces and pairs-of-earrings made respectively. There are two resource constraints: wire, of which there are only 100 feet, and gemstones, of which there are only 300. Thus the wire usage $4x + y$ cannot exceed 100, and the gemstone usage $3x + 4y$ cannot exceed 300. Finally, our profit function is $35x + 10y$, so our mathematical formulation is to maximize $35x + 10y$ subject to:

$$\begin{cases} 4x + y \leq 100 \\ 3x + 4y \leq 300 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

The first constraint has intercepts (25,0) and (0,100), while the second has intercepts (100,0) and (0,75). Plotting these on a graph gives us a feasible region.



Three of the corners of the region are obvious: (0,0), (25,0), and (0,75). However, the fourth is the intersection of the two constraint lines:

$$\begin{cases} 4x + y = 100 \\ 3x + 4y = 300 \end{cases}$$

By quadrupling the first equation and subtracting the second, we get $13x = 100$, so $x = \frac{100}{13} \approx 7.7$. Then $y = 100 - 4x$, so $y = 100 - \frac{400}{13} = \frac{900}{13} \approx 69.2$, giving (7.7,69.2) as the last corner point.

Finally, we test all four corners to see which maximizes the profit function:

$$\begin{aligned} \text{At } (0.0,\ 0.0) &: 35 \times 0.0 + 10 \times 0.0 = 0 \\ \text{At } (25.0,\ 0.0) &: 35 \times 25.0 + 10 \times 0.0 = 875 \\ \text{At } (0.0, 75.0) &: 35 \times 0.0 + 10 \times 75.0 = 750 \\ \text{At } (7.7, 69.2) &: 35 \times 7.7 + 10 \times 69.2 = 961.5 \end{aligned}$$

So our best strategy is to make an average of $\boxed{\text{7.7 necklaces and 69.2 pairs of earrings}}$ each week to realize a profit of $\boxed{\$961.54.}$

3. **(32 points)** *Consider the linear programming problem in which we maximize the profit function* $11x + 4y$ *subject to the conditions*

$$\begin{cases} 2x + y \leq 30 \\ 3x + 5y \leq 90 \\ 2x \leq 26 \\ x \geq 0, y \geq 0 \end{cases}$$

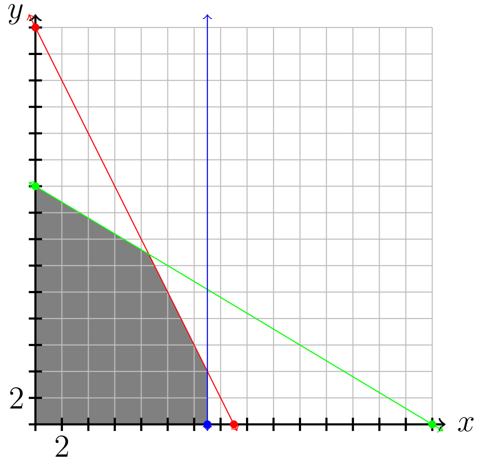
(a) **(10 points)** *Sketch the constraint lines, labeling the intercepts, on the axes below, and shade the feasible region (indicate the scale on the axes).*

The line $2x + y = 30$ has an x-intercept of $\frac{30}{2} = 15$ and a y-intercept of $\frac{30}{1} = 30$.

The line $3x + 5y = 90$ has an x-intercept of $\frac{90}{3} = 30$ and a y-intercept of $\frac{90}{5} = 18$.

The line $2x = 26$, since it has no y-components, is a vertical line, at the constant horizontal position of $x = 13$ (and specifically with the x-intercept of 13).

We can draw all of these on the axes by finding the intercepts and connecting them up with appropriate lines; we then shade the region under and to the left of these lines which is above and to the right of the axes:



(b) **(14 points)** *Find the coordinates for each potential feasible profit-maximizing point in the graph above.*

There are five corners to the region depicted in the graph, three of which are easily identified: the origin (0,0), the x-intercept (13,0), and the y-intercept (0,18). The other two points are the intersections of certain constraint lines: namely, the intersection of $2x+y=30$ with each of $2x=26$ and $3x+5y=90$.

For the first of these, we want to solve the simultaneous system of equations

$$\begin{cases} 2x + y = 30 \\ 2x \quad\;\; = 26 \end{cases}$$

Note that the second equation immediately gives us $x=13$, which we can plug into the first to solve for x:

$$\begin{aligned} 2 \cdot 13 + y &= 30 \\ 26 + y &= 30 \\ y &= \;\;4 \end{aligned}$$

so this point has coordinates (13,4).

The other point is at the intersection of the lines $2x+y=30$ and $3x+5y=90$ We thus want to solve the simultaneous system of equations

$$\begin{cases} 2x + \;\; y = 30 \\ 3x + 5y = 90 \end{cases}$$

which we can do by multiplying the former equation by 5 and subtracting the latter, leaving $7x=60$, so $x=\frac{60}{7}\approx 8.6$. To find y, we plug our known value of x back in to either equation and solve for y:

$$\begin{aligned} 2 \cdot \frac{60}{7} + y &= 30 \\ y &= 30 - 2 \cdot \frac{60}{7} = \frac{90}{7} \approx 12.9 \end{aligned}$$

so our coordinates for this point are about (8.6,12.9).

Our complete list of possible maximizing points is thus $\boxed{(0,0),\ (13,0),\ (13,4),\ (8.6,12.9),\ \text{and}\ (0,18).}$ If you like, you may drop (or not even consider in the first place) either or both of the points (0,0) or (13,0) as fundamentally inferior, leaving the shortened list $\boxed{(13,4),\ (8.6,12.9),\ \text{and}\ (0,18).}$

(c) **(8 points)** *Find the value of the pair* (x,y) *maximizing the profit on the above graph.*

With the three potential profit maximizers determined in the last part, we test each one to determine the actual profit:

$$\begin{aligned}
\text{At } (13.0,\ 4.0) &: 11 \times 13.0 + 4 \times \ \ 4.0 = 159 \\
\text{At } (\ \ 8.6, 12.9) &: 11 \times \ \ 8.6 + 4 \times 12.9 = 145.7 \\
\text{At } (\ \ 0.0, 18.0) &: 11 \times \ \ 0.0 + 4 \times 18.0 = \ \ 72
\end{aligned}$$

Thus, the first choice $\boxed{(13,4)}$ maximizes the profit.