Section 1.2: Simple Interest
Section 1.3: Annually Compounded Interest

MATH 105: Contemporary Mathematics

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Interest: the bedrock of finance

Often, an institution (or a person) needs to *use* more money than they *have*.

They could fill in the gap by getting an investor, or by *borrowing* money.

Outside of friendships or charity, the lending institution wants to be paid for this service.

Interest is the means by which a borrower pays for a loan.

You pay interest on a loan, but you also *earn* interest on a savings account, certificate of deposit, or government bond.

This is the same transaction flipped over: instead of paying a fee to a bank to borrow money, the bank pays you for money which you've loaned to them.

How much interest is right?

The amount of interest on a loan is dependant on three major features:

- The size of the loan
- The length of the loan
- Apparent risk
  - Economic factors
  - How likely the borrower is to not repay
  - What recourse the lender has if the borrower defaults
  - How much the lender can afford to lose the money

As a practical matter, all those "risk factors" get folded into a single number by the lender.

Types of loans

There are three common designs for how a loan (or a deposit, which is the same thing in reverse) can be structured.

Single-payment Money is provided (or deposited), and at a later date a larger sum of money is paid back (or withdrawn)
Examples: payday loans, certificates of deposit

Installment A large sum of money is provided (or invested), and periodically a much smaller sum of money is paid back (or paid out); the total of all the payments is larger than the original sum.
Examples: mortgages, car loans, annuities

Revolving Funds freely flow into or out of an account; interest is regularly calculated to modify the account balance.
Examples: savings accounts, credit cards

We will consider single-payment financial instruments in this chapter, and later visit installment structures in the next chapter.

Vocabulary for single-payment instruments

There are several technical terms in the language of finance; below are some which are particularly relevant to single-payment interest calculations.

Principal The amount loaned or deposited at the beginning of a financial scheme; it is also known as the *present value* and is usually denoted in calculations by the letter P.

Future value The amount repaid (or withdrawn) over the course of a financial scheme. In the case of a single-payment loan or deposit, this is just a single sum of money at the end. The future value is usually denoted by the letter F.

Total interest The difference between the future value and principal, usually denoted by the letter I.

Lifetime The lifetime of a financial scheme is the length of time between the initial transfer of money and the final transfer of money, and is denoted t.

Vocabulary examples

A simple single-payment loan

I offer to lend an acquaintance \$300, under the terms that he pays me back \$350 in 4 months.

In this scenario, we can attach values to the several terms of the loan:

- It has a *principal* or *present value* of \$300; in a calculation we would say $P = 300$.
- It has a *future value* of \$350; in a calculation we would say $F = 350$.
- The *total interest* on this loan is of \$50; in a calculation we would say $I = F - P = 50$.
- The *lifetime* of this loan is 4 months, or $\frac{4}{12}$ of a year; unless we are specifically working in months, years are the standard measurement of time in finance, so a typical calculation would use $t = \frac{4}{12} = \frac{1}{3}$.

Introducing simple interest

Recall that the interest charged on a loan is determined by the loan principal, the loan lifetime, and some nebulous “risk factor”.

As a practical matter, it makes sense for interest to be *proportional* to principal value and lifetime: so a $500 loan charges five times as much interest as a $100 loan, and a 3-year loan charges thrice as much interest as a one-year loan.

Thus, a straightforward calculation would be the following:

$$I = P \times t \times r$$

This formula contains the new value r, the “interest rate”, which is set by the bank as a rule, and described as an *annual percentage*.

The annual interest rate represents how much interest is charged *per year* on the loan, expressed *as a proportion of the loan principal*.

Calculating simple interest

$$I = P \times t \times r$$

An example problem

You have been offered a $200 loan charging *simple interest* at an annual interest rate of 3%, to be repaid in five years. How much would you have to pay back?

We can assign values to all the important numbers in the loan:

- Since $200 is being loaned, $P = 200$.
- Since the repayment is due in five years, $t = 5$.
- Since the interest rate is 3%, $r = 0.03$.

$$I = 200 \times 5 \times 0.03 = 30$$

But you have to pay back not just the interest of $30; you also need to pay back the principal. So the total to be repaid five years from now is **$230**.

Simple interest formulas

Determining interest and future value

$$I = Ptr \qquad F = P + Ptr = P(1 + tr)$$

Determining principal

$$P = \frac{I}{tr} \qquad P = \frac{F}{1 + tr}$$

Determining interest rate

$$r = \frac{I}{Pt} \qquad r = \frac{F - P}{Pt}$$

Determining lifetime

$$t = \frac{I}{Pr} \qquad t = \frac{F - P}{Pr}$$

Calculating principal

A known future value

A government bond matures to a value of \$50 after 30 years of earning 2% simple interest. How much does it originally cost to purchase the bond?

Here $F = 50$, $t = 30$, $r = 0.02$, and P is unknown, so

$$P = \frac{F}{1 + tr} = \frac{50}{1 + 30 \times 0.02} = \frac{50}{1.6} = 31.25$$

which means the bond is sold for **\$31.25** (and accrues \$18.75 in interest over those 30 years).

Calculating interest rate

Things I get in the mail

My bank offered a bonus of \$200 if I opened a savings account with \$15,000 and kept that balance for 6 months. What simple annual interest rate are they providing here?

Now $P = 15000$, $I = 200$, and $t = \frac{6}{12} = 0.5$; we want to know r.

$$r = \frac{I}{Pt} = \frac{200}{15000 \times 0.5} \approx 0.0267$$

so this offer is essentially providing a **2.67%** interest rate.

Calculating lifetime

Planning for the future

I have \$3000 to put in a certificate of deposit that pays 3.5% annual simple interest. How long should I leave it there in order to grow my investment to \$4000?

In this case $P = 3000$, $r = 0.035$, and $F = 4000$, so

$$t = \frac{F - P}{Pr} = \frac{4000 - 3000}{3000 \times 0.035} \approx 9.523$$

so my investment would have to sit there for at least **9.523 years**.

How simple interest is unfair

The formula for simple interest is based on the idea that earned interest should be proportional to the loan's lifetime—but this is an oversimplification!

A tale of two investments

Suppose Alice earns 2% annual simple interest on \$1000 for three years, while Bob earns 2% annual simple interest on three successive yearlong investments starting with \$1000 and reinvesting the proceeds of each.

It seems like these should be the same: same interest rate, same principal, same timeframe!

But if we calculate them out, then we get different results.

How simple interest is unfair (cont'd)

Alice's simple plan

Alice earns 2% annual simple interest on \$1000 for three years.

The future value is $F = \$1000 + \$1000 \times 0.02 \times 3 = \$1060$.

Bob's reinvestment scheme

Alice earns 2% annual simple interest on three consecutive one-year investments, starting with \$1000.

Bob's first year yields $F_1 = \$1000 + \$1000 \times 0.02 \times 1 = \$1020$.
His second investment earns simple interest on \$1020 for one year:

$$F_2 = \$1020 + \$1020 \times 0.02 \times 1 = \$1040.40.$$

And then that \$1040 is reinvested once more:

$$F_3 = \$1040.40 + \$1040.40 \times 0.02 \times 1 \approx \$1061.21.$$

So Bob's scheme earns \$1.21 more than Alice's!

Reinvestment for fun and profit

Alice's interest is calculated *exclusively* based on her principal.

Bob's interest is recalculated each year, based not only on principal but also on *previously earned interest*.

The process of earning interest on interest is called *compounding*, and a financial arrangement which uses interest earned each year to calculate the next year's is *annually compounding*.

Real-world interest calculations, both on savings and loans, make use of compounding.

Generalizing Bob's scheme

We saw how compounding could be worked out year-by-year for three years. But what if we had a timeframe of 10 or 30 years?

We would need some calculation which could skip that tedious process.

Let's take a look at how Bob's process worked but simplifying very little instead of performing calculations.

$$F_1 = 1000 + 1000 \times 0.02 \times 1 = 1000 \times 1.02$$
$$F_2 = F_1 + F_1 \times 0.02 \times 1 = 1000 \times 1.02 \times 1.02$$
$$F_3 = F_2 + F_2 \times 0.02 \times 1 = 1000 \times 1.02 \times 1.02 \times 1.02$$

So a pattern emerges!

Repeated multiplication simplified

For every year Bob has his \$1000 invested, its value would be multiplied by 1.02. So after, say, six years:

$$F = \$1000 \times 1.02 \times 1.02 \times 1.02 \times 1.02 \times 1.02 \times 1.02 \approx \$1126.16$$

Writing out all that multiplication is cumbersome, so we have a standard shorthand:

$$\underbrace{a \times a \times a \times \cdots \times a}_{n \text{ times}} = a^n$$

This operation is called *exponentiation*, and can also be written $a\hat{\ }y$. On a calculator, it is represented by a key labeled [^], [x^y], [y^x], or [$x^{\blacksquare}$].So the above calculation is simply:

$$F = \$1000 \times 1.02^6 \approx 1126.16$$

A general formula for annually compounding interest

Annually compounding interest with named parameters

We save (or borrow) a principal P, at an annual interest rate of r, for t years with annual compounding. What is our eventual balance F?

Every year applies interes by *multiplying* our balance by $1 + r$, so after t years:

$$F = P(1 + r)^t$$

This formula can be used to calculate future values on many of the same sort of financial instrument we looked at with simple interest!

Saving money

An example with a CD

Carla takes out a certificate of deposit, investing $3000 for seven years with a bank, for which the bank agrees to pay 2.6% compounding annually. What is her CD worth at the end? How much interest has she earned?

Using $P = 3000$, $r = 0.026$, and $t = 7$:

$$F = P(1+r)^t = 3000 \times 1.026^7 \approx 3590.48$$

so her CD is worth **$3590.48**. $3000 of that was her original investment and the rest is principal, so there is **$590.48** in earned interest.

Borrowing money

A single-payment loan

To buy a car, you borrow $5500 at an interest rate of 6%, to be repaid in full 8 years later. How much do you need to pay back in 8 years, and how much of that is interest?

Here $P = 5500$, $r = 0.06$, and $t = 8$:

$$F = P(1+r)^t = 5500 \times 1.06^8 \approx 8766.16$$

so you need to pay back **$8766.16**. You received $5500 in the first place which was the principal on the loan, so the interest here is **$3266.16**. Note that in this particular situation, you are paying a fairly large penalty in interest, and might want to reconsider the purchase!

Annual compounding vs. simple interest

The difference between the two schemes is often small, but over the long term, those small differences add up!

Hypothetical comparison

Consider a $400 investment at 3% annual interest. How would the balance differ over time using these two different interest calculations?