Section 1.4: Non-annual compounded interest

MATH 105: Contemporary Mathematics

University of Louisville

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Annual compounding, revisited

The idea behind annual compounding is that new interest is computed and added to the balance *each year*.

For a fixed-term multi-year deposit, this works, but what if we want to withdraw our money several months into a year?

One thing we could do differently is to compute a smaller chunk of interest more often.

Smaller interest, more often

A multiple-computation study

Suppose we want to compute and add in interest *quarterly* on a \$1000 balance with an annual interest rate of 5%, and want to know what the balance is after a full year.

Recall that for annual compounding we just did a *simple interest* calculation for each individual year. Now we do a simple interest calculation for each quarter (so $t = 0.25$):

$$F_1 = 1000.00 + 1000.00 \times 0.05 \times 0.25 = 1012.50$$
$$F_2 = 1012.50 + 1012.50 \times 0.05 \times 0.25 \approx 1025.16$$
$$F_3 \approx 1025.16 + 1025.16 \times 0.05 \times 0.25 \approx 1037.97$$
$$F_4 \approx 1037.97 + 1037.97 \times 0.05 \times 0.25 \approx 1050.95$$

so after a year the balance will be \$1050.95. Note that this is more than the nominal 5% per year in the interest rate!

Why compute interest more frequently?

There are two consequences of the calcluation we did in the last slide which are relevant:

- intermediary-stage values are now known; for instance, the balance halfway through the year was \$1025.16.
- the actual interest was higher than if it were compounded annually.

The first effect is undeniably good; the second maybe seems deceptive, but can be addressed with proper information.

Simplifying our calculations

Same study, but with less button-mashing

How can we simplify that calculation of quarterly interest on a $1000 balance with an annual interest rate of 5% for a full year?

Recall that the first calculation looked like this:

$$F_1 = 1000.00 + 1000.00 \times 0.05 \times 0.25 = 1012.50$$

which simplifies to $F_1 = 1000 \times (1 + 0.05 \times 0.25)$.

We want to apply that same multiplicative factor four times, so we might compute:

$$F = 1000 \times (1 + 0.05 \times 0.25)^4 \approx 1050.95$$

And for more emphasis on the "four quarters per year" aspect, we may write it as:

$$F = 1000 \times \left(1 + \tfrac{0.05}{4}\right)^4 \approx 1050.95$$

Applying our simplification

An extension of the last question

Suppose, as before, we want to compute and add in interest *quarterly* on a $1000 balance with an annual interest rate of 5%, but now we want to know what the balance is after 6 years.

As previously, we see that every quarter's interest application is a multiplication by $1 + \frac{0.05}{4}$. Six years measured in quarters is $6 \times 4 = 24$ quarters, so we want to perform that multiplication *twenty-four times*:

$$F = 1000 \times \left(1 + \frac{0.05}{4}\right)^{24} \approx 1347.35$$

for a final balance of $1347.35.

Building a formula

$$F = 1000 \times \left(1 + \frac{0.05}{4}\right)^{6\times4} \approx 1347.35$$

This calculation makes use of the principal $P = 1000$, the annual interest rate $r = 0.05$, and the lifetime $t = 6$, but it also uses a new quantity $n = 4$, the number of *compounding periods* per year.

Note that the expression $\frac{0.05}{4}$ is the *periodic* interest rate, i.e., the proportion of the balance returned in interest over a single compounding period, while 6×4 is the lifetime measured in compounding periods. This gives us the general formula:

$$F = P\left(1 + \frac{r}{n}\right)^{tn}$$

Sometimes the periodic interest rate is denoted by the letter $i = \frac{r}{n}$, and the number of compounding periods by $m = tn$.

Example calculations

Why stop at quarters?

I take out a \$500 loan whose annual interest rate of 18% is compounded *monthly*. How much would I need to pay it off after 9 months? After 2 years?

In both scenarios, $P = 500$, $r = 0.18$, and $n = 12$.
In the first scenario, since the lifetime was given in months, we could either establish $t = \frac{9}{12} = 0.75$ or, more straightforwardly, $m = 9$, so:

$$F = 500\left(1 + \frac{0.18}{12}\right)^{9} \approx 571.69$$

so I would have to pay back **\$571.69** (of which **\$71.69** is interest).
In the second scenario, $t = 2$, giving:

$$F = 500\left(1 + \frac{0.18}{12}\right)^{2\times12} \approx 714.75$$

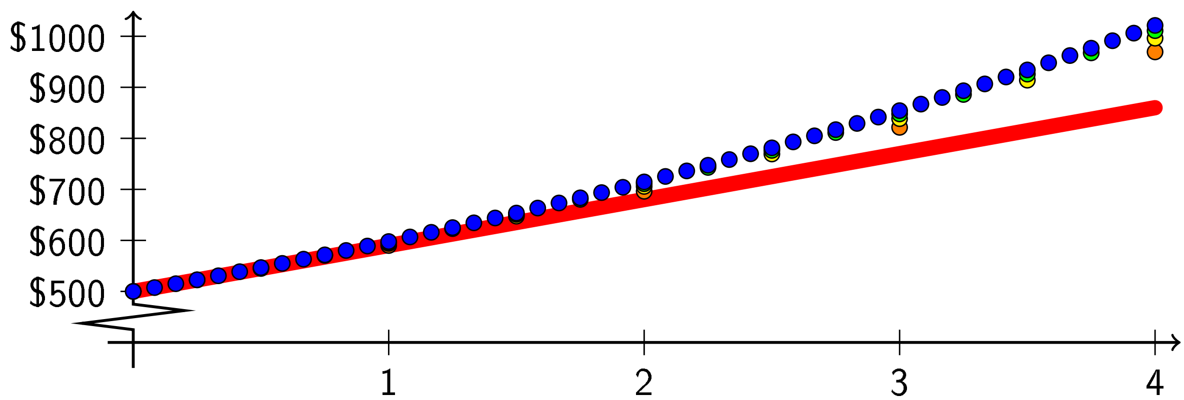
so I would have to pay back **\$714.75** (of which **\$214.75** is interest).

Variations in compounding periods

In general, more frequent compounding increases the long-term balance, but not by much!

Hypothetical comparison

Consider a \$500 loan with a 18% annual interest rate. How would the balance differ over 4 years using different compounding periods?



Taking it to the limit

Diminishing returns

How does a \$500 loan with a 18% annual interest rate for four years change as we increase the number of compounding periods?

As the last slide indicated, the returns on increasing compounding frequency decrease rapidly:

$$500\,(1+0.18)^4 \approx 969.39$$
$$500\left(1+\tfrac{0.18}{2}\right)^{4\times 2} \approx 996.28$$
$$500\left(1+\tfrac{0.18}{4}\right)^{4\times 4} \approx 1011.19$$
$$500\left(1+\tfrac{0.18}{12}\right)^{4\times 12} \approx 1021.74$$
$$500\left(1+\tfrac{0.18}{52}\right)^{4\times 52} \approx 1025.94$$
$$500\left(1+\tfrac{0.18}{365}\right)^{4\times 365} \approx 1027.03$$

Compounding continuously

When n is very large, the compounding becomes *continuous*.

There is a formula for what happens in this case too:

$$\text{As } n \text{ gets very large, } P\left(1+\frac{r}{n}\right)^{tn} \text{ approaches } Pe^{rt}$$

where $e \approx 2.718281828459$.

You won't be expected to work out continuous-compounding problems in this course, but knowing that there is a limiting behavior is useful!

Unveiling the truth

One *disadvantage* of nonannual compounding is that it conceals the truth: 5% annual rate compounded monthly isn't actually a 5% growth over a year!

A useful measure is the *annual percentage rate* (or *annual percentage yield*, which describes what percentage growth actually occurs yearly as a result of interest.

An APR example

If I borrow $1000 at 7% annual interest compounded monthly, what is the *actual* percentage growth after a year?

After one year, the future value is

$$F = 1000 \times (1+\frac{0.07}{12})^{12} \approx 1072.29.$$

so the growth percentage is $\frac{1072.29-1000}{1000} \approx 7.3\%$.

From the particular to the abstract

Our calculation in the last slide for the APR was

$$\frac{1000 \times (1 + \frac{0.07}{12})^{12} - 1000}{1000}$$

Here 1000 was the principal, 0.07 the annual interest rate, 12 the number of compounding periods per month, so in the abstract the APR is

$$\frac{P\left(1 + \frac{r}{n}\right)^n - P}{P} = \left(1 + \frac{r}{n}\right)^n - 1$$

Note that the amount and lifetime of the loan are not necessary to calculate an APR!

One interest rate, many annual percentages

Something as simple as a "5% annual interest rate" could mean many different things in different circumstances:

Compounded annually $(1 + \frac{0.05}{1})^1 - 1 = 5\%$ APR.
Compounded semiannually $(1 + \frac{0.05}{2})^2 - 1 = 5.0625\%$ APR.
Compounded quarterly $(1 + \frac{0.05}{4})^4 - 1 \approx 5.0945\%$ APR.
Compounded monthly $(1 + \frac{0.05}{12})^{12} - 1 \approx 5.1162\%$ APR.
Compounded weekly $(1 + \frac{0.05}{52})^{52} - 1 \approx 5.1246\%$ APR.
Compounded daily $(1 + \frac{0.05}{365})^{365} - 1 \approx 5.1267\%$ APR.
Compounded continuously $e^{0.05} - 1 \approx 5.1271\%$ APR.

All the formulas in one place

Annual compounding ($n = 1$):

$$F = P(1 + r)^t$$

Periodic compounding:

$$F = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$F = P\,(1 + i)^m \text{ where } i = \frac{r}{n} \text{ and } m = nt$$

$$\text{APR} = \left(1 + \frac{r}{n}\right)^n - 1$$

Continuous compounding:

$$F = Pe^{rt}$$

$$\text{APR} = e^r - 1$$