Sections 1.5 and 1.6: Other calculations on compounded interest

MATH 105: Contemporary Mathematics

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Periodic compounding

$$F = P\left(1 + \frac{r}{n}\right)^{nt}$$

We can use this formula to cover both periodic and annual compounding (just let $n = 1$ for the annual case).
There are five named values in here:

- P: principal a.k.a. present value or initial balance
- F: future value a.k.a. final balance
- r: annual interest rate
- t: lifetime of account
- n: number of compounding periods per year

Up until now, our formula has been set up to calculate F from the other four.

But there are other possible calculations!

Determining value of an unknown principal

Planning for the future

I would like to buy a \$7000 used car in five years. How much should I put into a 5% annual interest (compounding annually) CD to be able to afford this?

We know that this investment would involve annually compounding interest with a rate of $r = 0.05$ for a lifetime of $t = 5$ years:

$$F = P(1.05)^5$$

but here we know a desired value for F, not P:

$$7000 = P(1.05)^5$$

and we can solve for P with an appropriate division:

$$\frac{7000}{1.05^5} = P$$

so our initial principal should be $\frac{\$7000}{1.05^5} \approx \$5484.68$.

Generalizing the formula

We want to be able to compute P, given every other element of this formula:

$$F = P\left(1 + \frac{r}{n}\right)^{n \times t}$$

Since the right side is a multiplication, we can isolate one factor by *dividing by the other*.

$$\frac{F}{\left(1 + \frac{r}{n}\right)^{nt}} = P$$

One rule that's useful in mathematics is that $\frac{1}{x^k} = x^{-k}$, so we could also rewrite this as

$$P = F\left(1 + \frac{r}{n}\right)^{-nt}$$

which can be thought of as running time backwards, just using a negative "time" value to get from the future to the present instead of the other way around.

Using this formula

Taking out a pre-windfall loan

I stand to get \$10,000 in four years, but I need funds now. In expectation of being able to repay my loan with my windfall, how much could I borrow at a 7% annual rate compounding monthly?

Here I have $r = 0.07$ and $n = 12$, and hope to take out a principal which leads to a future value of $F = 10000$ in $t = 4$ years. We can use either of the two formulas on the previous slide.

$$P = \frac{10000}{\left(1 + \frac{0.07}{12}\right)^{4\times 12}} \text{ or } P = 10000\left(1 + \frac{0.07}{12}\right)^{-4\times 12}$$

getting a result of \$7563.99.

Another example usage of this formula

Calculating bond rates

Series Q bonds have an annual interest rate of 1.96% compounded quarterly. They are "fully matured" to their face value after 30 years. How much would it cost to purchase a bond with a face value of \$50?

The bond is described by the parameters $F = 50$, $t = 30$, $r = 0.0196$, and $n = 4$. We wish to know its fair present-day value P:

$$P = \frac{50}{\left(1 + \frac{0.0196}{4}\right)^{30\times 4}} \text{ or } P = 50\left(1 + \frac{0.0196}{4}\right)^{-30\times 4}$$

and with either calculation we get the present value of \$27.81.

The return of Sir Not-Appearing-in-this-Class

The same algebra can be used to convert the *continuous* compounding calcuation $F = Pe^{rt}$ to either of the forms

$$P = \frac{F}{e^{rt}} \text{ or } P = Fe^{-rt}$$

Real-world bond rates

Suppose that, as before, series Q bonds have an annual interest rate of 1.96% and fully mature to their face value after 30 years, but now compound continuously. How much would it cost to purchase a bond with a face value of $50?

As before, $F = 50$, $t = 30$, and $r = 0.0196$, and now we wish to know its fair present-day value P using a continuous-compounding formula:

$$P = \frac{50}{e^{0.0196\times 30}} \text{ or } P = 50e^{-0.0196\times 30}$$

and with either calculation we get the present value of $27.77.

Interest rates: the elephant in the room

The most significant feature of any loan or deposit is its *interest rate*.

Unscrupulous lenders often try to hide the interest rate with the favorable-looking parts of the terms.

The compounding period is often irrelevant, so we'll look at how we can calculate the *APR* (which essentially presumes annual compounding).

The scope of the problem

A sample question

An acquaintance offers you a five-year loan of \$700, after which you are required to pay back \$1000, while your bank is offering personal loans at an 8% APR. Which is better?

The question is: does that five-year loan you were offered have an APR higher or lower than 8%?
If we called its rate r, then we would have the following equation:

$$1000 = 700(1+r)^5$$

and we can solve for r by reversing the process of taking a fifth power!

$$(1+r)^5 = \frac{1000}{700}$$
$$1+r = \sqrt[5]{\frac{1000}{700}}$$
$$r = \sqrt[5]{\frac{1000}{700}} - 1 \approx 7.4\%$$

More details on reversing exponents

The process of reversing an exponent is known as a *root* or *radical*.

$$\text{If } x^n = y, \text{ then } x = \sqrt[n]{y}.$$

We read this as "the nth root of y".

In addition, there's a useful mathematical rule that $(x^a)^b = x^{ab}$, so that

$$\left(\sqrt[n]{x}\right)^n = x = x^1 = x^{\frac{1}{n}\cdot n} = \left(x^{1/n}\right)^n$$

and so another form for $\sqrt[n]{x}$ is $x^{1/n}$.

Roots can be computed on a calculator using a key typically labeled $\boxed{\sqrt[x]{\ }}$ or $\boxed{x^{1/y}}$.

$\sqrt[n]{x}$ is a messy and strange creature if x is negative; it never is in any of the cases we're looking at.

A general rate-calculation formula

None of the numbers we looked at in our last example were special, so we should be able to generalize.

Same question, but with named values

An account has a balance of P now, which will grow to F over the course of t years. What is the APR on this account?

As before, we can place these values into a relationship and solve for r:

$$F = P(1+r)^t$$
$$\frac{F}{P} = (1+r)^t$$
$$\sqrt[t]{\frac{F}{P}} = 1 + r$$
$$\sqrt[t]{\frac{F}{P}} - 1 = r \text{ or } r = \left(\frac{F}{P}\right)^{1/t} - 1$$

A sample of our rate-calculation formula

Account growth

I gave an example a while back about Chase offering me a \$200 bonus on a \$15,000 six-month deposit. What is the effective APR here?

Here $P = 15000$ and $I = 200$, so $F = 15200$, while $t = \frac{6}{12} = \frac{1}{2}$. So:

$$r = \sqrt[t]{\frac{F}{P}} - 1 = \sqrt[1/2]{\frac{15200}{15000}} - 1 \approx 2.68\%$$

Note we could alternatively calculate

$$r = \left(\frac{15200}{15000}\right)^{\frac{1}{1/2}} - 1 = \left(\frac{15200}{15000}\right)^2 - 1 \approx 2.68\%$$

The many things we can pull out of account information

Present value from future value

$P = \frac{F}{(1+\frac{r}{n})^{nt}}$ or $P = F(1+\frac{r}{n})^{-nt}$.

APR from compounding structure

$\text{APR} = \left(1+\frac{r}{n}\right)^n - 1$

APR from values and lifetime

$\text{APR} = \sqrt[t]{\frac{F}{P}} - 1$ or $\text{APR} = \left(\frac{F}{P}\right)^{1/n} - 1$

Lifetime from values and rate—sneak preview!

$t = \frac{\log \frac{F}{P}}{n \log\left(1+\frac{r}{n}\right)}$

What's this "log" thing? We'll see next time!