Section 1.7: Time-to-maturity calculations
Section 1.8: Inflation

MATH 105: Contemporary Mathematics

University of Louisville

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Knowing when you'll be done

We've seen how to work out most of the critical parameters of interest growth from all of the others, but there's one we haven't worked out yet.

A sensible question about investments

Eva has put \$3000 into a savings account earning 2.5% annual interest compounding annually. How many years will it take her savings to grow to \$5000?

We could do this with guess-and-correct, but that'd take a lot of tedious computations!

After 10 years she has $\$3000 \times 1.025^{10} \approx \$3840.25$ — not enough!
After 25 years she has $\$3000 \times 1.025^{25} \approx \$5561.83$ — too much!
After 20 years she has $\$3000 \times 1.025^{20} \approx \$4915.85$ — almost there!
After 21 years she has $\$3000 \times 1.025^{21} \approx \$5038.75$ — just right!
But this is an awful lot of computation. Can we simplify it?

A new algebraic question

In this particular scenario, we're trying to find the smallest value of t which satisfies

$$3000 \times 1.025^t \geq 5000$$

In practice, we could treat that as an equality, and just round to the nearest year. But how do we solve it?

$$1.025^t = \frac{5000}{3000}$$

The tool we will need to get further is the *logarithm*.

What is a logarithm?

The *common logarithm* of a number is the answer to the question "what power would we raise 10 to, in order to get this number?"

Example logarithms

The logarithm of 10,000 is 4, because $10^4 = 10000$.
The logarithm of 1 is 0, because $10^0 = 1$.
The logarithm of 0.001 is -3, because $10^{-3} = 0.001$.
The logarithm of 40 is a little more than 1.6, because $10^{1.6} \approx 39.81$.

The first three above are moderately straightforward, but the last would need a calculator to compute "log(40)"!

A lot of common measures are based on logarithms: the Richter earthquake scale measures the logarithm of vibration intensity, for instance.

But why should we use logarithms?

The logarithm has several extraordinary properties, one of which we're going to use:

$$\log(x^n) = n \log x$$

so applying a logarithm to an expression with an exponent magically converts the exponent to a multiplicative term, which we can work with. So if we wanted to solve for an exponent in the equation

$$x^n = y$$

we could take a logarithm of both sides

$$log(x^n) = \log y$$

and then use the above cool property.

So what about Eva?

Remembering our calculation at the beginning of the lesson, we got stuck at

$$1.025^t = \frac{5}{3}$$

Taking the logarithm of both sides, we get

$$\log(1.025^t) = \log \frac{5}{3}$$

And using our clever property on the left,

$$t \log 1.025 = \log \frac{5}{3}$$

And now all we need is to divide by this logarithm to get t alone:

$$t = \frac{\log \frac{5}{3}}{\log 1.025} \approx 20.687$$

which we can round up to 21 years, since compounding is annual.

Twice the logarithms means twice the fun

Calculators typically have two different logarithm keys, which produce different answers.
[log] is the "common logarithm", which determines the right exponent to place on the number 10. Scientists and engineers use this a lot.
[ln] is the "natural logarithm", which determines the right exponent to place on the number e (yup, the same one as in compound interest!). Mathematicians prefer this to the common logarithm.
You can use either of them as long as you're consistent! $\frac{\log(5/3)}{\log 1.025}$ and $\frac{\ln(5/3)}{\ln 1.025}$ both give the right answer before, but $\frac{\log(5/3)}{\ln 1.025}$ won't.

Also, on calculators which don't display functions onscreen, you press [log] *after* entering the number you want to take the logarithm of.

From the specific to the general

What sort of question might we want to answer *generally* about solving for time in interest calculations?

Eva's problem, generalized

If we have an account with initial principal P, subject to an annual interest rate r compounded n times per year, how long will it take the account to reach a balance of F?

As always, we'll start with the standard form of our interest formula:

$$F = P\left(1 + \frac{r}{n}\right)^{nt}$$

And in this particular case, we want to solve algebraically for t.

The most complicated slide of the day

$$F = P\left(1+\frac{r}{n}\right)^{nt}$$
$$\frac{F}{P} = \left(1+\frac{r}{n}\right)^{nt}$$
$$\log\frac{F}{P} = \log\left(1+\frac{r}{n}\right)^{nt}$$
$$\log\frac{F}{P} = tn\log\left(1+\frac{r}{n}\right)$$
$$\frac{\log\frac{F}{P}}{n\log\left(1+\frac{r}{n}\right)} = t$$

So now we have a general formula for t from other features! Sometime you want to know the number of compounding periods instead, which would be

$$m = nt = \frac{\log\frac{F}{P}}{\log\left(1+\frac{r}{n}\right)}$$

Note that it probably makes sense to round this up, since the number of compounding periods should be a whole number.

. . . and back to the specific

A sample interest-growth question

Ismail has \$2500 in a savings account earning 1.8% annual interest compounding quarterly. How long will it be until his savings have grown to \$3000?

Here we can just calculate the total number m of quarters taken, taking our present value to be $P = 2500$, desired future value $F = 3000$, annual interest rate $r = 0.018$, and number of periods per year $n = 4$:

$$m = \frac{\log\frac{F}{P}}{\log\left(1+\frac{r}{n}\right)} = \frac{\log\frac{3000}{2500}}{\log(1+\frac{0.018}{4})} \approx 40.607$$

which, since compounding happens over an integer number of quarters, means he needs **41 quarters**, or, alternatively, **10.25 years**.

Continuous interest calculations

If an account *continuously compounds*, then the time doesn't need to be a whole number of years, months, etc.

With an APR r, the calculation $F = P(1+r)^t$ yields

$$t = \frac{\log \frac{F}{P}}{\log(1+r)}$$

and with continuous compounding we don't need to round off.

If you're given a *continuous compounding rate* (which you won't, in this class), we can solve $F = Pe^{rt}$ to get $t = \frac{\ln \frac{F}{P}}{r}$.

Continuous compounding, continued

Watch out for loan growth!

If I have a continuously compounding \$1000 loan with an APR of 7.2%, how long will it take for the principal to reach \$1300?

Here we have an APR of 7.2%, so we can solve for the lifetime of the loan:

$$t = \frac{\log \frac{1300}{1000}}{\log 1.072} = 3.7736$$

so it would take about **3.7736 years** (note that we don't round off if the interest is continuous).

Quick mental math: the Rule of Seventy

Doubling time

How long would it take an account subject to an annual interest rate of r and continuously compounding to double in value?

Here we are solving for t in this equation:

$$2P = Pe^{rt}.$$

If we solve that, we get

$$t = \frac{\ln 2}{r} \approx \frac{0.693}{r}$$

The Rule of Seventy

To find the doubling time of a continuously compounding account, divide 70 (or 69.3) by the nominal interest percentage.

For instance, a continuously compounding account with 4% interest would double in about $\frac{70}{4} = 17.5$ years.

Putting it all together

For annual compounding, $t = \frac{\log\left(\frac{F}{P}\right)}{\log(1+r)}$, rounded up.

For periodic compounding, $m = \frac{\log\left(\frac{F}{P}\right)}{\log\left(1+\frac{r}{n}\right)}$, rounded up, and then $t = \frac{m}{n}$.

For continuous-compounding given by an annual percentage rate, $t = \frac{\log\left(\frac{F}{P}\right)}{\log(1+\text{APR})}$, leaving a fractional part on your answer.

For continuous compounding with a nominal annual rate continuously compounded, $t = \frac{\ln\left(\frac{F}{P}\right)}{r}$.

What is inflation?

Inflation is when the apparent value of money decreases, and as a result things become more expensive.

Inflation is not necessarily a bad thing! Generally, it is good for borrowers, bad for lenders, and stimulates spending.

Most economies are given to natural inflation, both as a result of increasing supply of money and various changes in supply and demand of products.

How do we measure inflation?

Since inflation represents a decrease in the purchasing power of a unit of currency, we can *observe* inflation by noting how prices change.

However, tracking the price of one good, while subject to inflation, may be much more affected by the properties of that good, and when it becomes scarce or in extreme demand.

Typically economists measure inflation by selecting a “market basket” of goods and tracking the price of that over time.

Those numbers in the newspaper

In the US, one of the standard market-basket price measurements is the Consumer Price Index (CPI), calculated by the Bureau of Labor Statistics.

This economic indicator gives a sense of how the cost of things have grown (and how the dollar has shrunk in value) over time.

For instance, the CPI in November 1974 had a value of 51.0; by September 1986 it had a value of 110.0.Thus, over these 12 years the dollar halved in purchasing power.

Using CPI to time-shift a quantity of money

If you read a historical story in which someone gets, say, \$200 in 1950, those stories will usually tell you how much that is in present-day dollars. With a CPI, we can do that ourselves.

"The past is a foreign country" —L.P. Hartley

The average family income in 1960 was \$6,691. How much is that in 2017 dollars, using the knowledge that the CPI in January 1960 was 29.41, and the CPI in early 2017 was 147.98?

We might determine how many market baskets the average income in 1960 could buy: $\frac{\$6691}{\$29.41} \approx 227.507$.
This same number of market baskets, in 2017, costs $227.507 \times \$147.98 \approx \$33666.58$, so that average 1960 income corresponds to a **\$33,666.58** income in 2017 dollars.

CPI formulas in general

The method used in the last slide can be generalized. To convert "year A dollars" to "year B dollars":

$$\text{Value in Year B} = \frac{\text{Year B CPI}}{\text{Year A CPI}} \times \text{Value in Year A}$$

Because of that we call the *percentile growth* $\frac{\text{Year B CPI}}{\text{Year A CPI}} - 1$ the *inflation percentage*. For a single year, we can call it the *inflation rate*.

Calculating inflation rates

The early-2009 CPI is 131.825; the early-2010 CPI is 133.733. What was the inflation rate for 2009?

Here we calculate $\frac{133.733}{131.825} - 1 \approx 1.45\%$.