Section 2.1: Intro to Installment Structures and Investment Plans

MATH 105: Contemporary Mathematics

University of Louisville

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What is an installment plan?

An installment plan is when some transaction on a financial structure (deposit, withdrawl, payment, etc.) is done regularly over time.

Installment plans allow an enormous effect—beyond the scope of most people's personal finances—to be produced with a large number of much smaller contributions, or alternatively to parcel a huge financial instrument into more manageable pieces.

Types of installment plans

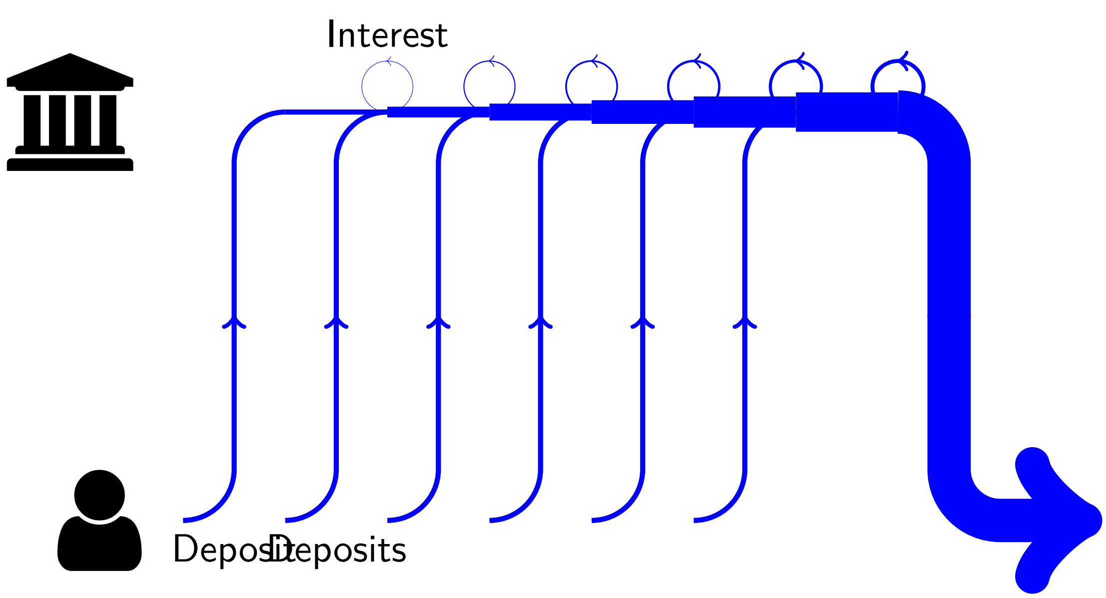
There are three common installment plans which are regular occurrances:

Planned investment A payment is made regularly into a savings account (or similar), and after several years there is a large quantity of money available.

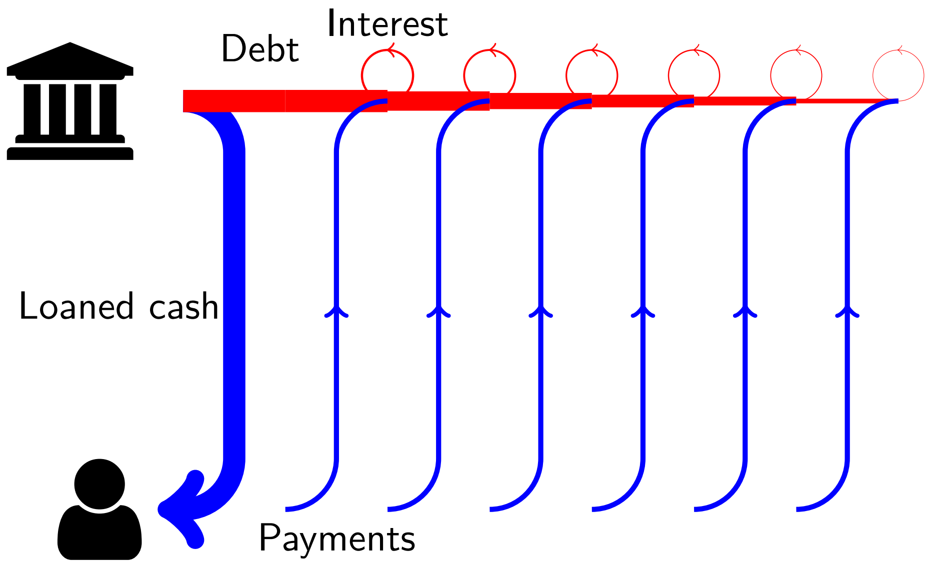
Installment loan A bank provides a large sum of money up front, and the borrower pays off the loan with regular small payments over several years.

Annuity A large deposit is made into a savings account (or similar), and then the depositor draws off a small sum periodically for several years.

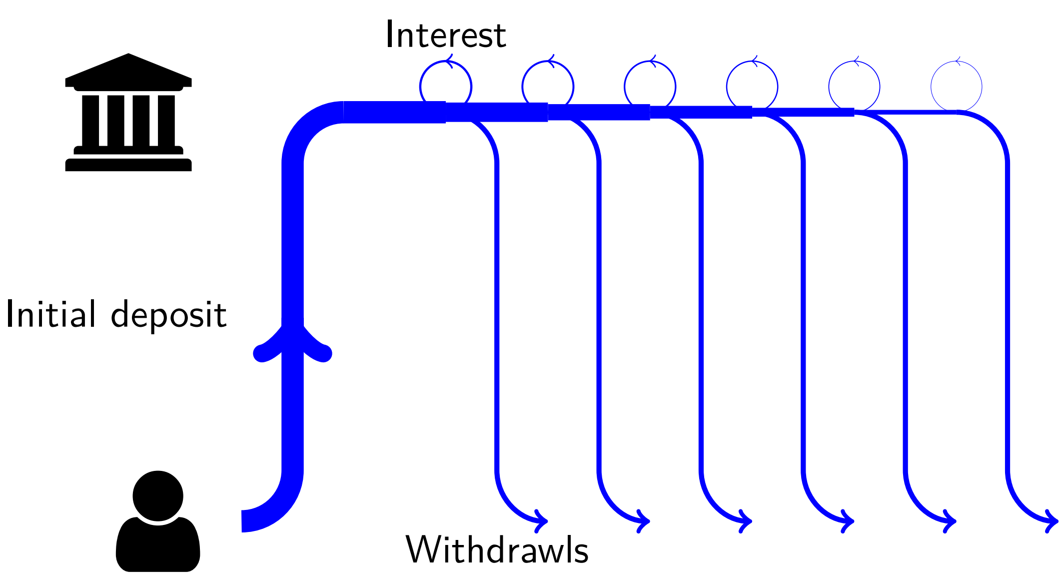
Planned investments, visualized



Installment loans, visualized



Annuities, visualized



Finding the size of a planned investment

Let's look at a simple plan, small enough to be worked out by hand:

A short, simple investment plan

I put \$100 per year into an initially empty account which pays 4% annual interest compounded annually, at the end of a year, for 4 years. What's the value of my account at the end of this program?

We can look at what happens year by year:

Year	Starting balance	Interest	Contribution	Ending Balance
1	\$ 0.00	\$ 0.00	\$100.00	\$100.00
2	\$100.00	\$ 4.00	\$100.00	\$204.00
3	\$204.00	\$ 8.16	\$100.00	\$312.16
4	\$312.16	\$12.49	\$100.00	\$424.65

Obviously, this won't scale up to a large installment plan!

Slicing up our calculation differently

Instead of considering the balance at each phase, let's consider what happens to each chunk of money.

The \$100 contribution at the end of the first year undergoes three years of interest to have a future value of $\$100 \times 1.04^3 \approx 112.49$.

The \$100 contribution at the end of the second year undergoes two years of interest to have a future value of $\$100 \times 1.04^2 = 108.16$.

The \$100 contribution at the end of the third year undergoes one year of interest to have a future value of $\$100 \times 1.04 \approx 104.00$.

The \$100 contribution at the end of the fourth year undergoes no interest; its future value is just its present value $\$100.00$.

Add up these contributions, and we get

$$\$112.49 + \$108.16 + \$104 + \$100 = \$424.65!$$

Making that calculation even simpler

So the answer we wanted was just

$$100 \times 1.04^3 + 100 \times 1.04^2 + 100 \times 1.04 + 100.$$

We can of course factor out a multiple of 100:

$$100 \times (1.04^3 + 1.04^2 + 1.04 + 1),$$

and then, we can do a neat trick with a clever multiplication by 1:

$$100 \times (1.04^3 + 1.04^2 + 1.04 + 1) \times \frac{1.04 - 1}{1.04 - 1} = 100 \times \frac{1.04^4 - 1}{1.04 - 1}$$

And this is really pretty easy to calculate, and would be that way even with more than 4 years!

Generalizing our scenario

A general form for investment plans

We contribute A dollars each compounding period for m periods to an account which earns interest at a periodic rate i. How much has accumulated at the end?

Our first contribution earns $m - 1$ periods worth of interest, our second $m - 2$, and so forth until we get no interest (0 periods worth) on our last contribution.
Total value at the end is thus

$$F = A(1 + i)^{m-1} + A(1 + i)^{m-2} + \cdots + A(1 + i)^2 + A(1 + i) + A$$

or factoring out A,

$$F = A\left((1 + i)^{m-1} + (1 + i)^{m-2} + \cdots + (1 + i)^2 + (1 + i) + 1\right)$$

Generalizing our scenario (continued)

$$F = A\left((1+i)^{m-1} + (1+i)^{m-2} + \cdots + (1+i) + 1\right)$$
$$F = A\left((1+i)^{m-1} + (1+i)^{m-2} + \cdots + (1+i) + 1\right)\frac{(1+i)-1}{(1+i)-1}$$
$$F = A\frac{(1+i)^m - 1}{i}$$

If you are given annual information, you may have r and t instead of m, so:

$$F = A\frac{\left(1+\frac{r}{n}\right)^{nt} - 1}{r/n} = An\frac{\left(1+\frac{r}{n}\right)^{nt} - 1}{r}$$

Applications

An involved application

I put \$20 at the end of each month into a savings account which earns 2.5% annual income compounding monthly. How much will I have after 4 years, and how much will I have earned in interest?

This particular scenario has contributions of $A = 20$ and interest at a monthly rate of $i = \frac{0.025}{12}$ occurring over $m = 4 \times 12 = 48$ months, so the total accumulation is

$$F = 20 \times \frac{\left(1+\frac{0.025}{12}\right)^{48} - 1}{\frac{0.025}{12}} \approx 1008.54$$

We actually supplied most of the resulting **\$1008.54** ourselves! A monthly contribution of \$20 for 48 months is a total of \$960, so only **\$48.54** is interest.

Everything on one place

Total accumulation

$$F = An\frac{\left(1+\frac{r}{n}\right)^{nt}-1}{r} \text{ or } F = A\frac{(1+i)^{nt}-1}{i}$$

Total contribution

Total of all deposits is Am or Ant

Total interest

Earned interest is $An\left(\frac{\left(1+\frac{r}{n}\right)^{nt}-1}{r}-t\right)$ or $A\left(\frac{(1+i)^{nt}-1}{i}-m\right)$