Section 2.2: Amortized Loans and Annuities

MATH 105: Contemporary Mathematics

University of Louisville

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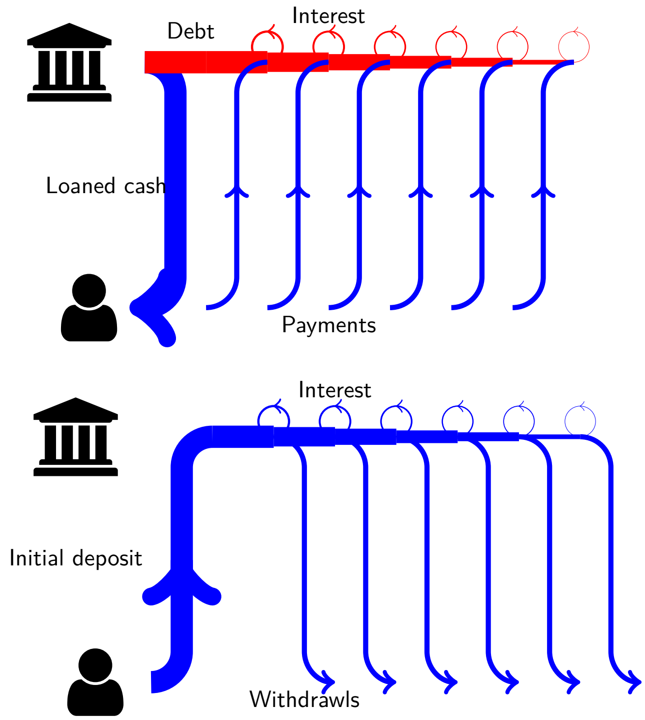
The fundamental concept

Amortization is a term of art for reducing something by prorating its cost over time.

An *amortized loan* is one in which a large debt assumed up-front is paid back over time.

An *annuity* is a somewhat different transfer with similar underlying mathematics, where a large up-front deposit to a savings account is depleted slowly over time.

Amortized loans vs. annuities



How to calculate them

The good news is that amortized loans and annuities are the same concept, so, for instance, these two questions have the same answer.

A loan-principal question

You can afford to pay back \$200 per month out of your household budget for 4 years on a loan with an annual interest rate of 5%, compounding monthly. How much could you borrow?

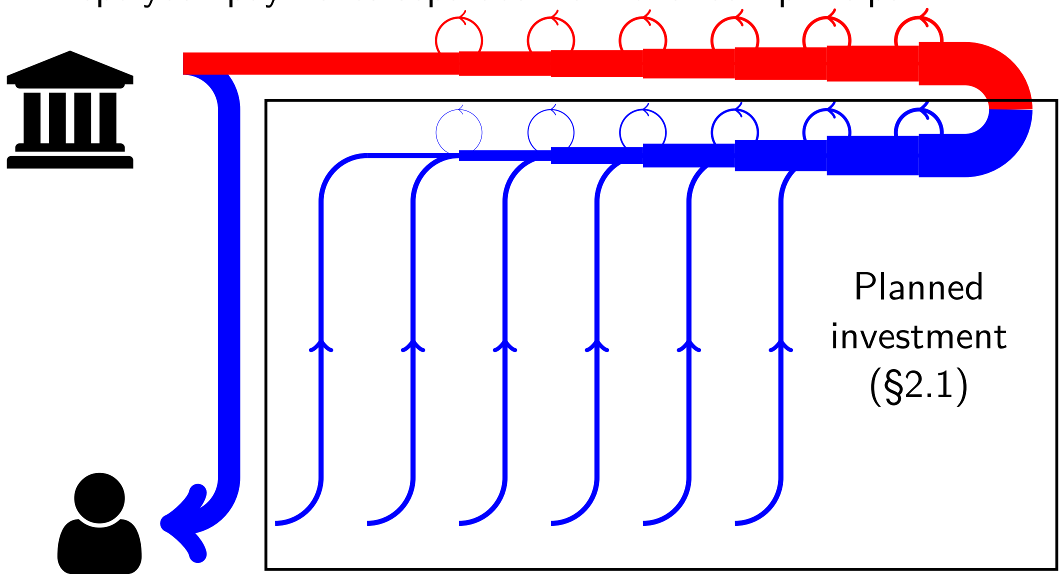
An annuity question

You want to set up an annuity which pays out \$200 per month for 4 years, using an account with an annual interest rate of 5%, compounding monthly. How much do you need to put in the account to set up this annuity?

In both cases you seek a *present value* based on the size of future installments and interest rates.

A familiar picture, an unfamiliar setup

Here's another view of what an installment loan might look like, if the bank kept your payments separate from the loan principal:



The moral of the story

So the present value of a loan needs to be whatever *would* mature at the end of the loan period to equal the value of an investment plan defined by the payments given.

Recall this example:

A loan-principal question

You can afford to pay back $200 per month out of your household budget for 4 years on a loan with an annual interest rate of 5%, compounding monthly. How much could you borrow?

Here, we want a loan principal which after 4 years has grown to exactly equal the value of an investment plan investing $200 per month, with both accounts growing at 5% annually with monthly compounding.

Finally, some arithmetic!

A loan-principal question

You can afford to pay back \$200 per month out of your household budget for 4 years on a loan with an annual interest rate of 5%, compounding monthly. How much could you borrow?

Let's compute the value of 48 months of monthly investment of \$200 with an annual interest rate of 5% (see §2.1):

$$F = A\frac{(1+i)^m - 1}{i} = 200 \times \frac{\left(1+\frac{0.05}{12}\right)^{48} - 1}{\frac{0.05}{12}} \approx 10602.98$$

and we can consider our loan a five-year single-payment loan whose future value *should* be \$10602.98, and we need to know present value (see §1.5):

$$P = \frac{F}{(1+i)^m} = \frac{10602.98}{\left(1+\frac{0.05}{12}\right)^{48}} \approx 8684.59$$

And our final analysis?

A loan-principal question

You can afford to pay back \$200 per month out of your household budget for 4 years on a loan with an annual interest rate of 5%, compounding monthly. How much could you borrow?

We saw on the last slide that we could borrow \$8684.59.

We also can see that we make 48 repayments of \$200, or \$9600 in total.

Thus, we have, over time, repaid the entire principal and \$915.41 in *interest*.

Can we generalize?

Let's look at that same problem with named instead of numeric parameters:

The generalized question

We can repay an amount A every period for m periods, towards a loan with periodic interest rate i. What is the quantity P that we can borrow?

Our repayment has a total value (cf. §2.1) of

$$A\frac{(1+i)^m - 1}{i}.$$

If the loan principal was untouched for the same length of time, it would have a value of (cf. §1.4):

$$P(1+i)^m$$

These values should be the same, so the loan "cancels out".

Solving for P

We require thus that

$$P(1+i)^m = A\frac{(1+i)^m - 1}{i}$$

and want to solve for P, so we divide both sides by $(1+i)^m$.

$$P = A\frac{(1+i)^m - 1}{(1+i)^m i}$$

This permits cancellation, if not very nice cancellation, between the numerator and denominator:

$$P = A\frac{1-(1+i)^{-m}}{i}$$

and so we have a final formula, usable for loans (or annuities).

Using our formula

A simple annuity

You want to set up a modest annuity paying \$500 each quarter for 10 years. You have access to an account paying 3% annual interest compounding quarterly. What should you put in the account to start the annuity up?

Here $A = 500$, $n = 4$, $t = 10$, and $r = 0.03$; you can calculate $m = 40$ and $i = 0.0075$ too (or not).

$$P = 500 \times \frac{1 - \left(1 + \frac{0.03}{4}\right)^{-4\times 10}}{\frac{0.03}{4}} \approx \$17223.47$$

so the startup cost would be \$17223.47.

Since it pays out $\$500 \times 40 = \$20000$, it earns \$2776.53 in interest.

Everything on one slide

Initial principal of a loan/annuity

$$P = A\frac{1 - (1+i)^{-m}}{i} \text{ or } P = P = A\frac{1 - \left(1 + \frac{r}{n}\right)^{-n\times t}}{\frac{r}{n}}$$

Total payment/withdrawl

Total of all installments is Am or Ant

Total interest

Earned interest is total of all installments minus principal:

$$A\left(m - \frac{1 - (1+i)^{-m}}{i}\right) \text{ or } A\left(nt - \frac{1 - \left(1 + \frac{r}{n}\right)^{-n\times t}}{\frac{r}{n}}\right)$$