Section 2.4: Amortization schedules and payoff amounts

MATH 105: Contemporary Mathematics

University of Louisville

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Visualizing an Amortized Loan

Way back on September 7th, we interactively worked out what happens after every year of an investment plan.

We can do something similar for an amortized loan, which gives us a structure called an *amortization schedule*.

A short-term amortized loan

I've borrowed \$3000 for six quarters on a quarterly repayment plan with 8% annual interest compounded quarterly. How does the repayment work out in practice?

First, let's figure out how big the repayment is:

$$A = \frac{Pi}{1-(1+i)^{-m}} = \frac{3000 \times 0.02}{1-1.02^{-6}} \approx 535.58$$

Now we can chart out the loan's whole lifetime.

Amortization Schedules

A short-term amortized loan, summarized

We borrow \$3000 for six quarters, with a \$535.58 repayment and 2% interest each quarter.

Quarter	Initial balance	Amount paid	Interest charged	Principal repaid	Final balance
1	\$3000.00	\$ 535.58	\$ 60.00	\$ 475.58	\$2524.42
2	\$2524.42	\$ 535.58	\$ 50.49	\$ 485.09	\$2039.33
3	\$2039.33	\$ 535.58	\$ 40.79	\$ 494.79	\$1544.54
4	\$1544.54	\$ 535.58	\$ 30.89	\$ 504.69	\$1039.85
5	\$1039.85	\$ 535.58	\$ 20.80	\$ 514.78	\$ 525.07
6	\$ 525.07	\$ 535.57	\$ 10.50	\$ 525.07	\$ 0.00
Total		\$3213.47	\$213.47	\$3000.00	

Fundamental calculation tools for amortization

- The start-of-period balance each year is the end-of-year balance from the previous period.
- The interest charged each period is the *periodic* interest rate times the starting balance.
- The principal paydown is the total payment minus interest payment.
- The end-of-year balance is the start-of-year balance minus the principal paydown.
- The total of all the principal repayments equals the starting principal.
- The total amount of all payments is the total paid.
- The total amount of all interest charges is the total interest or *finance charge*.

Per-period balances without tables

Most of the entries in our table are pretty straightforward to calculate from limited information. The one thing we can't calculate easily (without using the table!) is how the balance changes.

A loan too big for tables

I have a 30-year mortgage at an annual rate of 5% with an initial principal of \$120,000. Payments and interest are monthly. How much balance will still be left on the loan after ten years?

The answer isn't two-thirds of the original balance, even though it seems like it should be!

What we are seeking here is called the *payoff amount*, i.e., how much I would need to pay off the entire loan partway through its life.

Working backwards with what we already know

A loan too big for tables

I have a 30-year mortgage at an annual rate of 5% with an initial principal of \$120,000. Payments and interest are monthly. How much balance will still be left on the loan after ten years?

Let's start by finding out what my monthly payment would be ($P = 120000$, $t = 30$, $n = 12$, $r = 0.05$):

$$A = \frac{Pi}{1-(1+i)^{-m}} = \frac{120000 \times \frac{0.05}{12}}{1-\left(1+\frac{0.05}{12}\right)^{-360}} \approx 644.19$$

Now, what I have left, ten years into the loan, is a *twenty-year* loan with this payment, and we can use the formula in §2.2 for it:

$$P = A\frac{1-(1+i)^{-m}}{i} = 644.19 \times \frac{1-\left(1+\frac{0.05}{12}\right)^{-240}}{\frac{0.05}{12}} \approx 97610.48$$

Generalizing the formula

Same loan, but without numbers

I have a loan of $\$P$ being paid back over m periods with periodic interest rate i. How much of the balance remains after m_0 periods?

Our periodic payment is

$$A = \frac{Pi}{1-(1+i)^{-m}}$$

And then what remains after m_0 periods is a loan with periodic payment A and a lifetime of $m - m_0$ periods, so our balance at this time would be

$$P_{m_0} = A \times \frac{1-(1+i)^{-(m-m_0)}}{i} = P \times \frac{1-(1+i)^{m_0-m}}{1-(1+i)^{-m}}$$

Seeing that in context

Let's look at the result of that formula on our original 5% mortgage, over its whole lifetime:

$$P_{m_0} = 120000 \times \frac{1-\left(1+\frac{0.05}{12}\right)^{m_0-360}}{1-\left(1+\frac{0.05}{12}\right)^{-360}}$$