Section 2.6: Lifetime of installment systems

MATH 105: Contemporary Mathematics

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Relevant formulas

Recall that, depending on the type of financial instrument, one of two formulas is centrally important:

$$F = A\frac{(1+i)^m - 1}{i}$$

$$P = A\frac{1-(1+i)^{-m}}{i}$$

We've already solved both for A (see §2.3).

Solving for i is unfortunately very difficult.

But one thing worth looking at is solving for m (or t, given $m = nt$)!

A case study

Building up a fortune

I want to become a millionaire by depositing \$2000 each quarter into a money-market account which earns 3.8% annual interest compounding quarterly. How long will this take?

Here we have an investment plan (§2.1), with desired future value $F = 1000000$, interest rate $r = 0.038$, compounding $n = 4$ times per year, and quarterly contribution $A = 2000$.

The thing we don't know, and want to find out, is m, in this equation:

$$1000000 = 2000 \times \frac{\left(1 + \frac{0.038}{4}\right)^m - 1}{\frac{0.038}{4}}$$

Oh, no, more algebra!

$$1000000 = 2000 \times \frac{\left(1 + \frac{0.038}{4}\right)^m - 1}{\frac{0.038}{4}}$$

$$4.75 = \frac{1000000}{2000} \times \frac{0.038}{4} = \left(1 + \frac{0.038}{4}\right)^m - 1$$

$$5.75 = 1.0095^m$$

$$\log(5.75) = \log\left(1.0095^m\right)$$

$$\log(5.75) = m \log(1.0095)$$

$$184.9995 \approx \frac{\log(5.75)}{\log(1.0095)} = m$$

so our task will take (rounded up) **185 quarters** or **46.25 years**.

Same thing but with an annuity

Whatcha gonna do when the lake runs dry?

I have a \$70,000 annuity earning 3.2% annual interest compounded monthly. I'd like to take out \$800 each month to supplement my income. How long can I afford to do so?

Here we have not an accumulative investment but an *annuity*, described by this equation from §2.2:

$$P = A\frac{1-(1+i)^{-m}}{i}$$

We know $A = 800$, $P = 70000$, $r = 0.032$, and $n = 12$. We want to find m (or t).

$$70000 = 800\frac{1-\left(1+\frac{0.032}{12}\right)^{-m}}{\frac{0.032}{12}}$$

Return of the Planet of the Algebra

$$\begin{aligned}
70000 &= 800\frac{1-\left(1+\frac{0.032}{12}\right)^{-m}}{\frac{0.032}{12}} \\
\frac{2.8}{12} = \frac{70000}{800} \times \frac{0.032}{12} &= 1-\left(1+\frac{0.032}{12}\right)^{-m} \\
\left(1+\frac{0.032}{12}\right)^{-m} &= 1-\frac{2.8}{12} \\
\log\left(1+\frac{0.032}{12}\right)^{-m} &= \log\left(1-\frac{2.8}{12}\right) \\
-m\log\left(1+\frac{0.032}{12}\right) &= \log\left(1-\frac{2.8}{12}\right) \\
m &= \frac{\log\left(1-\frac{2.8}{12}\right)}{-\log\left(1+\frac{0.032}{12}\right)} \approx 99.77
\end{aligned}$$

So it ends after **99 full monthly payments and one short payment**, or **8.25 years of full payments** with one last, short payment.

How do we generalize?

We have two separate equations, as mentioned previously; in both, we want to solve for m (or $t = \frac{m}{n}$):

$$F = A\frac{(1+i)^m - 1}{i} \qquad P = A\frac{1-(1+i)^{-m}}{i}$$

$$\frac{Fi}{A} = (1+i)^m - 1 \qquad \frac{Pi}{A} = 1-(1+i)^{-m}$$

$$\frac{Fi}{A} + 1 = (1+i)^m \qquad (1+i)^{-m} = 1 - \frac{Pi}{A}$$

$$\log\left(\frac{Fi}{A}+1\right) = \log(1+i)^m \qquad \log(1+i)^{-m} = \log\left(1-\frac{Pi}{A}\right)$$

$$\log\left(\frac{Fi}{A}+1\right) = m\log(1+i) \qquad -m\log(1+i) = \log\left(1-\frac{Pi}{A}\right)$$

$$\frac{\log\left(\frac{Fi}{A}+1\right)}{\log(1+i)} = m \qquad m = \frac{\log\left(1-\frac{Pi}{A}\right)}{-\log(1+i)}$$

Real-world situations

Getting out from under your debt

If you took out a 30-year, 5% mortgage with a \$100,000 principal, how much could you save overall by paying \$1000 per month instead of the bank-mandated payment?

First off, what *is* that bank-mandated payment?

$$A = \frac{Pi}{1-(1+i)^{-m}} = \frac{100000 \times \frac{0.05}{12}}{1-\left(1+\frac{0.05}{12}\right)^{-360}} \approx 536.82$$

so, paying \$536.82 each month for 360 months will result in a total payment of \$193,255.78, or a finance charge of \$93,255.78.

Real-world situations, continued

Getting out from under your debt

If you took out a 30-year, 5% mortgage with a $100,000 principal, how much could you save overall by paying $1000 per month instead of the bank-mandated $536.82 payment?

So now, we want to determine the value of m—which should be less than 360—if $A = 1000$.

$$m = \frac{\log\left(1 - \frac{P}{A} i\right)}{-\log(1+i)} = \frac{\log\left(1 - \frac{100000}{1000} \times \frac{0.05}{12}\right)}{-\log\left(1 + \frac{0.05}{12}\right)} \approx 129.62$$

so this increased payment brings the lifetime of the loan down to 130 months, nearly a third of the original!
And as a result the total paid is also smaller: 129.62847 payments of $1000 is a total of $129,628.47, with a finance charge of only $29,628.47.

Debt overpayment benefit visualized

Below is the potential of overpayment for this $100,000, 5% annual rate, 30-year mortgage.