Section 3.1: Introduction to Linear Programming

MATH 105: Contemporary Mathematics

University of Louisville

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Maximization and minimization

Many problems we encounter in daily life involve either *maximization* of something desirable or *minimization* of something undesirable.

- I have $20 in my pocket. How can I have the *most* fun at a street festival?
- I need to get to Shelbyville and Hurstborne. What combination of buses and biking segments gets me there in the *least* time?
- I have a lot of concert tickets to sell. How do I price them to make the *most* money?

Such problems are known as *optimization* problems.

Optimization is really hard!

Some of the most complicated and infamous problems in mathematics are questions of optimization. Two are particularly the subject of ongoing research.

The Traveling Salesman Problem

You want to go on a tour visiting each of several cities exactly once, and then return to your home cities. Distances among all the cities are known. What route minimizes the total distance traveled?

The Knapsack Problem

You have a backpack of limited carrying capacity, and you have several objects of varying values and bulk which you want to put into your backpack. How do you select objects in order to maximize the value of your haul, while staying under capacity limits.

These are *not* the sort of problems we'll be looking at, although some of the same tools we will learn can be applied to them.

How do we make optimization easy?

Every optimization problem has three basic elements:

- **Variables** The factors we're allowed to change: e.g. which routes we took when planning a trip, which things we bought at a festival, etc.
- **Objective** The thing we seek to maximize or minimize: e.g. minimizing travel time, maximizing value, etc.
- **Constraints** The rules on what values our variables can take on, e.g. we could only spend as much money as we have, or can only take a leg of a trip that connects to other legs of the trip.

We're going to look at a very limited set of optimization problems called *linear programs*.

In these problems, both the objective and constraint involve *linear* functions of the variables.

What does that kind of problem look like?

A prototypical linear programming problem

I am making Italian food to sell, and I have 50 tomatoes, 20 balls of mozzarella, 10 rounds of pizza dough, and 10 pounds of pasta. A caprese salad uses one tomato and one ball of mozarella, and sells for \$7. A pizza uses two tomatoes, one ball of mozarella, and a round of dough, and sells for \$11. A dish of spaghetti alla marinara uses three tomatoes and a third of a pound of pasta, and sells for \$10. What food should I make to maximize my revenue?

There's a lot to unpack in that example, but it has all three things we expect to find in an optimization problem.

What does that kind of problem look like?

Same problem, with numbers filed off

We have limited quantities of tomatoes, mozzarella, dough, and pasta. We can make capreses, pizzas, and spaghetti. We want to maximize our revenue.

Variables We have freedom to make whichever quantity of each foodstuff we want.

Objective We want to maximize the value of the food we've made.

Constraints We have limited quantities of each ingredient, which restrict the amount we could make of each food.

Successfully identifying all of these features will help us to build a *mathematical abstraction* of the problem.

The first aspect: Variables

Still trying to make this dinner work

I am making Italian food to sell, and I have 50 tomatoes, 20 balls of mozzarella, 10 rounds of pizza dough, and 10 pounds of pasta. A caprese salad uses one tomato and one ball of mozarella, and sells for $7. A pizza uses two tomatoes, one ball of mozarella, and a round of dough, and sells for $11. A dish of spaghetti alla marinara uses three tomatoes and a third of a pound of pasta, and sells for $10. What food should I make to maximize my revenue?

Above we have identified our three products, and we countrol how much or little we make of each. Let's give them names.

- We call the number of caprese salads x.
- We call the number of pizzas y.
- We call the number of dishes pf spaghetti z.

The second aspect: Objectives

Now we have variables in our system

I have 50 tomatoes, 20 balls of mozzarella, 10 rounds of pizza dough, and 10 pounds of pasta. I'll make x caprese salads, and each one uses one tomato and one ball of mozarella, and sells for $7. I make y pizzas, each of which uses two tomatoes, one ball of mozarella, and a round of dough, and sells for $11. I also make z dishes of spaghetti alla marinara, with each using three tomatoes and a third of a pound of pasta, and selling for $10. What food should I make to maximize my revenue?

Here we highlighted our goal (revenue maximization) and things relevant to that goal (how much money we get from each product).

- Making x salads earns us $7x$ dollars.
- Making y pizzas earns us $11y$ dollars.
- Making z pastas earns us $10z$ dollars.

Thus, we want to maximize the total revenue $7x + 11y + 10z$!

The third aspect: Constraints

Variables and an arithmetic expression

I have 50 tomatoes, 20 balls of mozzarella, 10 rounds of pizza dough, and 10 pounds of pasta. I'll make x caprese salads, and each one uses one tomato and one ball of mozarella. I make y pizzas, each of which uses two tomatoes, one ball of mozarella, and a round of dough. I also make z dishes of spaghetti alla marinara, with each using three tomatoes and a third of a pound of pasta. How do we maximize $7x + 11y + 10z$?

There are four different raw materials which we have a limited supply of. They've been labeled here in four colors, and each constrains us.

- z pastas use $\frac{1}{3}z$ pounds of pasta. We only have 10 pounds.
- y pizzas use y rounds of dough. We only have 10 rounds.
- x salads and y pizzas use $x + y$ mozzarellas. We only have 20.
- Our products use $x + 2y + 3z$ tomatoes and we only have 50.

Each of these dictates a *constraint inequality*.

The third aspect: Constraints, continued

- z pastas use $\frac{1}{3}z$ pounds of pasta. We only have 10 pounds.
- y pizzas use y rounds of dough. We only have 10 rounds.
- x salads and y pizzas use $x + y$ mozzarellas. We only have 20.
- Our products use $x + 2y + 3z$ tomatoes and we only have 50.

Let's write those as inequalities:

- $\frac{1}{3}z \leq 10$.
- $y \leq 10$.
- $x + y \leq 20$.
- $x + 2y + 3z \leq 50$.

We have one more non-obvious set of constraints: we can't make a negative number of anything!

- $x \geq 0$, $y \geq 0$, $z \geq 0$.

From description to abstraction

Our question in words

I have 50 tomatoes, 20 balls of mozzarella, 10 rounds of dough, and 10 lb. of pasta. A caprese uses 1 tomato and 1 ball of mozarella, and sells for \$7. A pizza uses 2 tomatoes, 1 ball of mozarella, and a round of dough, and sells for \$11. A dish of spaghetti uses 3 tomatoes and $\frac{1}{3}$ lb. of pasta, and sells for \$10. What food should I make?

Our question in equations

Maximize $7x + 11y + 10z$ subject to:

$$\begin{cases} \frac{1}{3}z \leq 10 \\ y \leq 10 \\ x + y \leq 20 \\ x + 2y + 3z \leq 50 \\ x, y, z \geq 0 \end{cases}$$