Section 3.1: Feasibility and Slack

MATH 105: Contemporary Mathematics

University of Louisville

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A quick recap

Problem 3.1.18

A nut company sells three blends: regular, deluxe, and supreme, with profits of \$4/pound on regular and deluxe and \$6/pound on supreme. The regular mix is 50% peanuts, 30% cashews, and 20% hazelnuts; deluxe is 30% peanuts, 40% cashews, and 30% hazelnuts; supreme is 20% peanuts, 40% cashews, and 40% hazelnuts. They have 800 lb. of peanuts, 400 lb. of cashews, and 295 lb. of hazelnuts. How do they maximize profit?

- Variables: x, y, and z lbs. of regular, deluxe, and supreme.
- Objective: Maximize $4x + 4y + 6z$.
- Constraints:
  - Peanuts: $0.50x + 0.30y + 0.20z \leq 800$
  - Cashews: $0.30x + 0.40y + 0.40z \leq 400$
  - Hazelnuts: $0.20x + 0.30y + 0.40z \leq 295$
  - Non-negativity: $x, y, z \geq 0$

The same problem expressed mathematically

Problem 3.1.18 (answer)

Maximize $4x + 4y + 6z$ subject to the conditions

$$\begin{cases} 0.50x + 0.30y + 0.20z \leq 800 \\ 0.30x + 0.40y + 0.40z \leq 400 \\ 0.20x + 0.30y + 0.40z \leq 295 \\ x, y, z \geq 0 \end{cases}$$

So, what can we *do* with this system?

In time, we're going to solve it.

For now, we can explore the notions of *feasibility* and *slack*.

Feasible and nonfeasible solutions

A production plan is *feasible* when it satisfies all the constraints, and *infeasible* when it does not.

Feasibility is the answer to the question of whether a particular production plan can be put into effect at all, or whether there aren't enough resources to do it.

Note that feasibility says nothing about whether a production plan is actually *good*; producing nothing at all is feasible, but it's not profitable.

Feasibility examples

Maximize $4x + 4y + 6z$ subject to the conditions

$$\begin{cases} 0.50x + 0.30y + 0.20z \leq 800 \\ 0.30x + 0.40y + 0.40z \leq 400 \\ 0.20x + 0.30y + 0.40z \leq 295 \\ x, y, z \geq 0 \end{cases}$$

Is it *feasible* to produce 400 pounds of regular, 300 pounds of deluxe, and 300 pounds of supreme? If so, how profitable is it?
We check each condition when $x = 400$, $y = 300$, and $z = 300$:

$$\begin{cases} 0.50 \times 400 + 0.30 \times 300 + 0.20 \times 300 = 350 \leq 800 \\ 0.30 \times 400 + 0.40 \times 300 + 0.40 \times 300 = 360 \leq 400 \\ 0.20 \times 400 + 0.30 \times 300 + 0.40 \times 300 = 290 \not\leq 295 \end{cases}$$

so this production plan is unfeasible.

Feasibility examples, continued

Maximize $4x + 4y + 6z$ subject to the conditions

$$\begin{cases} 0.50x + 0.30y + 0.20z \leq 800 \\ 0.30x + 0.40y + 0.40z \leq 400 \\ 0.20x + 0.30y + 0.40z \leq 295 \\ x, y, z \geq 0 \end{cases}$$

Is it *feasible* to produce 300 pounds of regular, 250 pounds of deluxe, and 200 pounds of supreme? If so, how profitable is it?
We check each condition when $x = 300$, $y = 250$, and $z = 200$:

$$\begin{cases} 0.50 \times 300 + 0.30 \times 250 + 0.20 \times 200 = 265 \leq 800 \\ 0.30 \times 300 + 0.40 \times 250 + 0.40 \times 200 = 270 \leq 400 \\ 0.20 \times 300 + 0.30 \times 250 + 0.40 \times 200 = 215 \leq 295 \end{cases}$$

so it is feasible, and the profit is

$$\$4 \times 300 + \$4 \times 250 + \$6 \times 200 = \$3400$$

Slack

Let's notice something about the constraints in that last example:

$$\begin{cases} 0.50 \times 300 + 0.30 \times 250 + 0.20 \times 200 = 265 \leq 800 \\ 0.30 \times 300 + 0.40 \times 250 + 0.40 \times 200 = 270 \leq 400 \\ 0.20 \times 300 + 0.30 \times 250 + 0.40 \times 200 = 215 \leq 295 \end{cases}$$

Our solution not only satisfies all three constraints, but satisfies them all to a with room to spare!

The extent to which a constraint is *oversatisfied* (i.e., the difference between the left and right side) is called *slack*.

For instance, the above production schedule has a slack of 535 pounds of peanuts, 130 pounds of cashews, and 80 pounds of hazelnuts.

Interpreting slack

Slack is the material left over after you perform a production plan.

When you have several raw materials, you're likely to have slack in at least one of them.

However, if you have slack in *every* constraint, then you know you could be producing more of something.

For instance, the production of 300 pounds of regular, 250 of deluxe, and 200 of supreme above is definitely not our best production plan.

Picking up the slack

$$\begin{cases} 0.50 \times 300 + 0.30 \times 250 + 0.20 \times 200 = 265 \leq 800 \\ 0.30 \times 300 + 0.40 \times 250 + 0.40 \times 200 = 270 \leq 400 \\ 0.20 \times 300 + 0.30 \times 250 + 0.40 \times 200 = 215 \leq 295 \end{cases}$$

Our heaviest constraint is on hazelnuts (only 80 pounds of them left) and we use 0.4 pounds of hazelnuts per pound of supreme, so we can use 80 pounds more hazelnuts by increasing z by $\frac{80}{0.4} = 200$.
Let's look at what happens with $x = 300$, $y = 250$, and $z = 400$:

$$\begin{cases} 0.50 \times 300 + 0.30 \times 250 + 0.20 \times 400 = 305 \leq 800 \\ 0.30 \times 300 + 0.40 \times 250 + 0.40 \times 400 = 350 \leq 400 \\ 0.20 \times 300 + 0.30 \times 250 + 0.40 \times 400 = 295 \leq 295 \end{cases}$$

This is also more *profitable*:

$$\$4 \times 300 + \$4 \times 250 + \$6 \times 400 = \$4600.$$

Solutions without slack

So is $x = 300$, $y = 250$, $z = 400$ our best solution?

Nope; there are other ways to improve, but they'd require us to *reduce* production of one thing while increasing production of another, so they're complicated.

So bringing a slack quantity down to zero improves our plan, but it doesn't make it the best possible.

However, the fact that optimal solutions have zero slack in at least one constraint will become useful later.

Another example

Question 3.1.16

A farmer has a 200-acre farm, and plans to plant oats and/or soybeans. Oats require 4 pounds of seed and 3 workdays per acre, while soybeans require 5 pounds of seed and 2 workdays per acre. The labor pool provides 570 workdays, and the largest possible seed delivery is 920 pounds. If oats provide a profit of \$150 per acre and soybeans \$200 per acre, how does the plan to plant 50 acres of oats and 130 of soybeans look?

- Variables: x and y acres of oats and soybeans respectively.
- Objective: Maximize $150x + 200y$.
- Constraints:
  - Land: $x + y \leq 200$
  - Seed: $4x + 5y \leq 920$
  - Labor: $3x + 2y \leq 570$
  - Non-negativity: $x, y \geq 0$

Another example, continued

Question 3.1.16, mathematicalized

Maximize $150x + 200y$ subject to the conditions $x, y \geq 0$ and

$$\begin{cases} x + y \leq 200 \\ 4x + 5y \leq 920 \\ 3x + 2y \leq 570 \end{cases}$$

How well does $x = 50$ and $y = 130$ work?

We check feasibility:

$$\begin{cases} 50 + 130 = 180 \leq 200 \\ 4 \times 50 + 5 \times 130 = 850 \leq 920 \\ 3 \times 50 + 2 \times 130 = 410 \leq 570 \end{cases}$$

It's feasible with profit $\$150 \times 50 + \$200 \times 130 = \$33500$, but slack is suboptimal.

Another example, continued (continued)

$$\begin{cases} \phantom{4\times{}} 50 + \phantom{5\times{}} 130 = 180 \leq 200 \\ 4 \times 50 + 5 \times 130 = 850 \leq 920 \\ 3 \times 50 + 2 \times 130 = 410 \leq 570 \end{cases}$$

Our slack of 20 acres suggests we might as well plant more acres of something.
Increasing y by 20 would use too much seed, but increasing y by only $\frac{70}{5} = 14$ is safe and improves our profit:

$$\begin{cases} \phantom{4\times{}} 50 + \phantom{5\times{}} 144 = 194 \leq 200 \\ 4 \times 50 + 5 \times 144 = 920 \leq 920 \\ 3 \times 50 + 2 \times 144 = 438 \leq 570 \end{cases}$$

$$\$150 \times 50 + \$200 \times 144 = \$36300$$