Section 3.2: Graphing constraints

MATH 105: Contemporary Mathematics

University of Louisville

October 5, 2017

A problem from last time

Question 3.1.16, mathematicalized

Maximize $150x + 200y$ subject to the conditions $x, y \geq 0$ and

$$\begin{cases} x + y \leq 200 \\ 4x + 5y \leq 920 \\ 3x + 2y \leq 570 \end{cases}$$

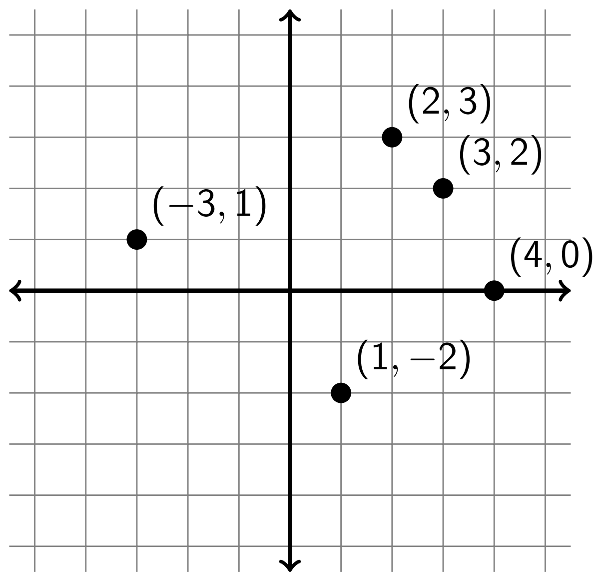
What is described here is a number of relationships, and a function to maximize, on *two variables*.

Systems of two variables can be visualized straightforwardly by using the coordinate plane.

We will use a visual representation to get a feel for exactly which production plans are *feasible*.

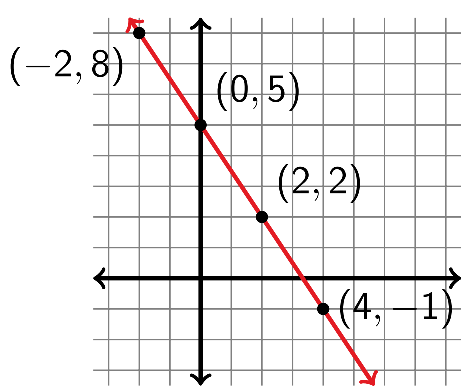
How coordinates work

Each point in the coordinate plane can be represented by an *ordered pair* (x, y): its horizontal position, followed by its vertical position.



Coordinatizing an equation

On the coordinate plane, what does an expression like $3x + 2y = 10$ mean?
We might note that there are a lot of ordered pairs which satisfy this equation, like $(2, 2)$, $(0, 5)$, and other unusual ones like $(4, -1)$ and $(-2, 8)$.
Let's plot those pairs:



All these points lie on a common *line*.

Linear equations

In general, equations of the form $ax + by = c$ are called *linear*, because their graphs are lines.

These are exactly the sort of expressions which (along with their related inequalities) appear in *linear programming*.

Understanding what these lines look like is integral to being able to interpret where a set of constraints is feasible.

Plotting lines in two easy steps

Fortunately, lines are uniquely determined by any two points.

Typically, the easiest points to find are called the *x-intercept* and *y-intercept*.

These are the answers resectively to the question "what point has a zero y-component" and "what point has a zero x-component".

If we have an equation $ax + by = c$, then this is usually a matter of simple algebra.

Returning to our original example

An equation we've seen before

How do we plot $3x + 2y = 10$?

We will compute the *intercepts*:

$$\begin{aligned} 3x_0 + 2 \times 0 &= 10 \\ 3x_0 &= 10 \\ x_0 &\approx 3.33 \\ (3.33, 0) \end{aligned} \qquad \begin{aligned} 3 \times 0 + 2y_0 &= 10 \\ 2y_0 &= 10 \\ y_0 &= 5 \\ (0, 5) \end{aligned}$$

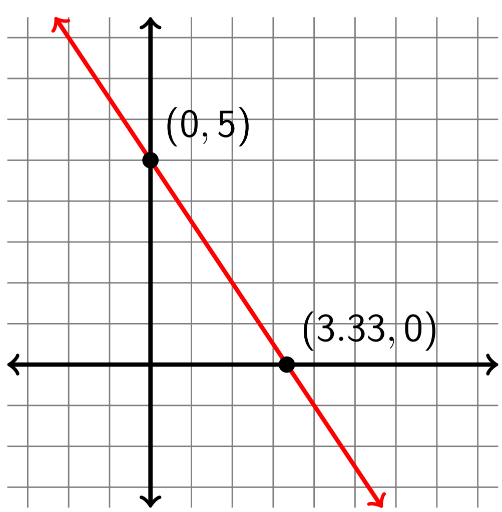
So our line is through $(3.33, 0)$ and $(0, 5)$.

Drawing that information

An equation we've seen before

How do we plot $3x + 2y = 10$?

We know this equation is of a line through $(3.33, 0)$ and $(0, 5)$:



Drawing lines form formulas

We can generalize the intercept calculation for a line $ax + by = c$:

$$\begin{aligned} ax_0 + b \times 0 &= c \\ ax_0 &= c \\ x_0 &= \tfrac{c}{a} \\ (\tfrac{c}{a}, 0) \end{aligned} \qquad \begin{aligned} a \times 0 + by_0 &= c \\ by_0 &= c \\ y_0 &= \tfrac{c}{b} \\ (0, \tfrac{c}{b}) \end{aligned}$$

and then draw a line between these two points.

Drawing several lines (intercept calculations)

Lines with equations from 3.1.16

Let's plot all three of these equations on the same plane:

$$\begin{aligned} x + y &= 200 \\ 4x + 5y &= 920 \\ 3x + 2y &= 570 \end{aligned}$$

We need intercepts for each equation:

$$(\tfrac{200}{1}, 0) = (200, 0) \qquad (0, \tfrac{200}{1}) = (0, 200)$$

$$(\tfrac{920}{4}, 0) = (230, 0) \qquad (0, \tfrac{920}{5}) = (0, 184)$$

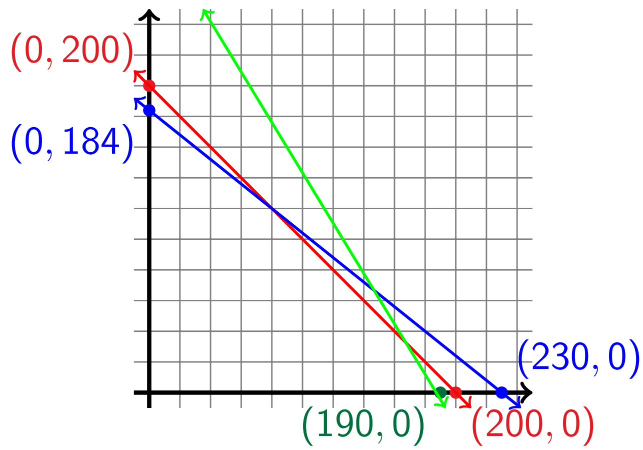
$$(\tfrac{570}{3}, 0) = (190, 0) \qquad (0, \tfrac{570}{2}) = (0, 285)$$

Drawing several lines (drawing to scale)

Lines with equations from 3.1.16

We are drawing lines through the following pairs of points: $(200, 0)$ to $(0, 200)$; $(230, 0)$ to $(0, 184)$; $(190, 0)$ to $(0, 285)$.

Let's establish a scale larger than what we've used; maybe each tick on our graph will have a length of 20 instead of 1.



Two special cases

There are two specific lines which only have one of the two intercepts:

$$ax + 0y = c \qquad 0x + by = c$$

The first line is *vertical* passing through $(\frac{c}{a}, 0)$.

The second line is *horizontal* passing through $(0, \frac{c}{b})$.

Trying to calculate the other intercept will result in a division by zero, so you'll know it's a special case.

Feasibility regions given constraints

Each constraint in a linear programming problem limits the answer to the region *below/to the left of* the line the constraint describes (except for the non-negativity conditions).

Thus, the region which is below and to the left of the constraint lines, and above and to the right of the coordinate axes, is called the *feasibility region*.

Feasibility regions visualized

Question 3.2.18

What are the feasible solutions to the constraints $2x + y \leq 30$, $x + y \leq 20$, $3x + 5y \leq 90$, $2x \leq 26$, $x \geq 0$, and $y \geq 0$?