Section 3.2: Finding Optimal Solutions

MATH 105: Contemporary Mathematics

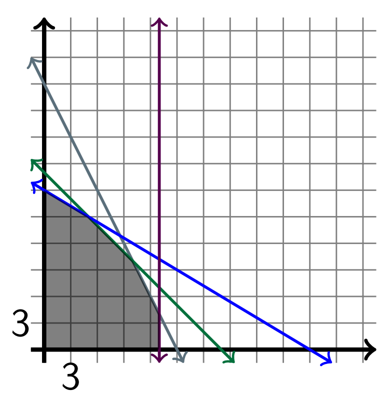
University of Louisville

October 11, 2017

Feasibility regions

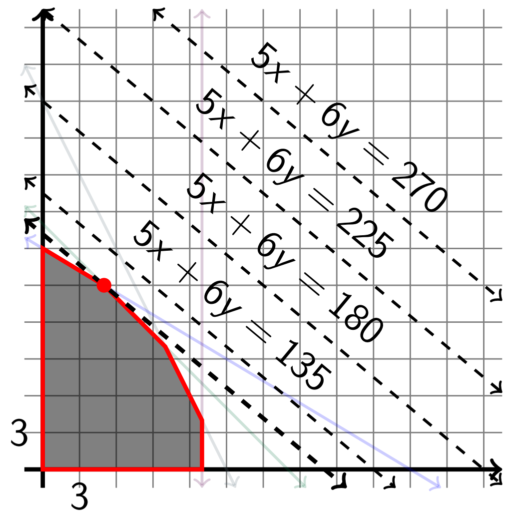
A familiar question (3.2.26)

Maximize $5x + 6y$ subject to $x, y \geq 0$ and $\begin{cases} 2x + y \leq 30 \\ x + y \leq 20 \\ 3x + 5y \leq 90 \\ 2x \leq 26 \end{cases}$



Profit curves

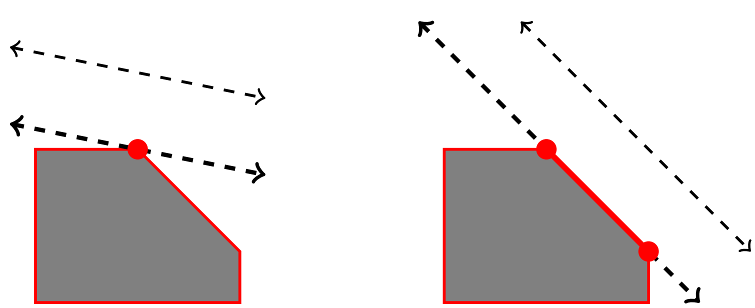
We want to find where in the shaded region $5x + 6y$ is maximized. Let's look at several lines corresponding to certain values for $5x + 6y$.



As these lines edge closer and closer to the feasibility region, there's some magical point where they touch!

The Cornerpoint Principle

As we slide a line towards a region with straight sides, there are two possibilities:



It can intersect the region in a single corner point...
...or in an edge, together with two corner points.

The Cornerpoint Principle

Any linear objective function is maximized at at least one corner of the feasible region.

Making use of the Cornerpoint Principle

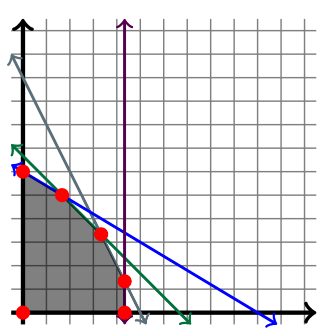
So now we know that *whatever* our objective is, it's optimized at some corner of the feasible region.

Our solution method will thus be: find the coordinates of all corners of the feasible region, and see which gives the highest objective value.

Unfortunately, many of the corners may require algebra to find!

Looking at a feasible region

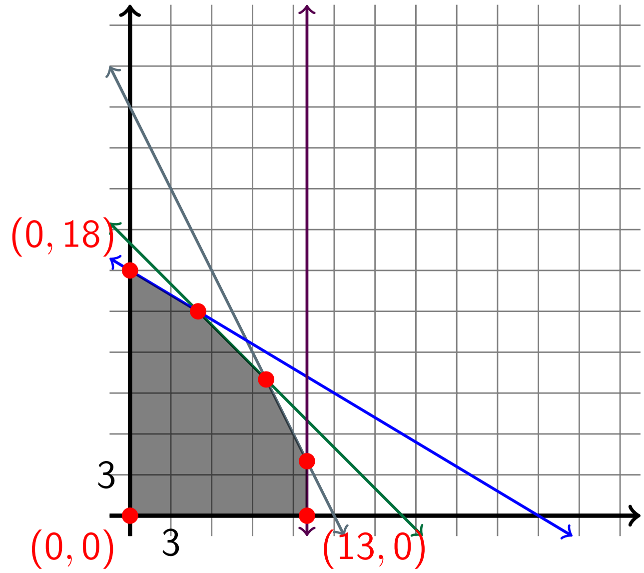
From an old familiar example: $\begin{cases} 2x + y \leq 30 \\ x + y \leq 20 \\ 3x + 5y \leq 90 \\ 2x \leq 26 \end{cases}$



Corners appear at the origin, the y-intercept of the blue line, the intersection point of the blue and green lines, the intersection of the green and grey, the intersection of the grey and purple, and the x-intercept of the purple.

Low-hanging fruit

$$\begin{cases} 2x + y \leq 30 \\ x + y \leq 20 \\ 3x + 5y \leq 90 \\ 2x \leq 26 \end{cases}$$

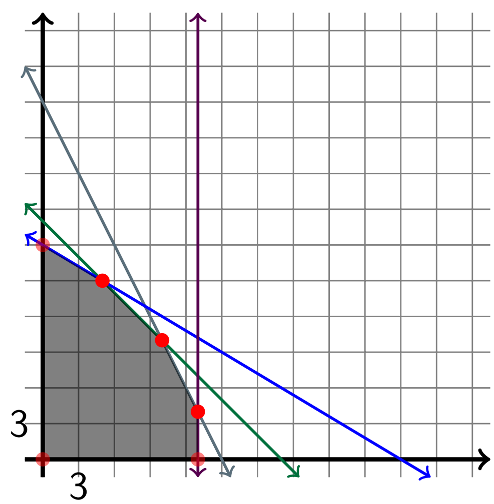


Three of our points are the origin and two intercepts.

The last three points will be harder to find.

Those more difficult points

$$\begin{cases} 2x + y \leq 30 \\ x + y \leq 20 \\ 3x + 5y \leq 90 \\ 2x \leq 26 \end{cases}$$



The interesting points are at the intersections of pairs of lines, so we need to solve the following three systems of equations:

$$\begin{cases} x + y = 20 \\ 3x + 5y = 90 \end{cases} \qquad \begin{cases} 2x + y = 30 \\ x + y = 20 \end{cases} \qquad \begin{cases} 2x + y = 30 \\ 2x = 26 \end{cases}$$

The easiest of the three intersections

Let's try to satisfy both of these equations simultaneously:

$$\begin{cases} 2x + y = 30 \\ 2x = 26 \end{cases}$$

From the second equation, $x = 13$.

Thus, y must satisfy $2 \times 13 + y = 30$, so:

$$y = 30 - 2 \times 13 = 4$$

Note that indeed $2 \times 13 + 4 = 30$ and $2 \times 13 = 26$, satisfying both equations.

And so we've got one of our three corner points: $(13, 4)$.

A harder intersection

Let's try to satisfy both of these equations simultaneously:

$$\begin{cases} 2x + y = 30 \\ x + y = 20 \end{cases}$$

Here we don't have any single equation telling us immediately what x or y is.
However, both equations have a single "y" in them, so we can subtract the second equation from the first to get:

$$(2x + y) - (x + y) = 30 - 20$$

or

$$x = 10$$

Then, we can substitute that into the second equation to get:

$$10 + y = 20$$

so $y = 10$ too, and $(10, 10)$ is where these two lines intersect.

And harder still!

$$\begin{cases} x + y = 20 \\ 3x + 5y = 90 \end{cases}$$

Now we don't know either value immediately, nor do we have an easy subtraction.

But we can scale one of these equations up by a factor of 3:

$$\begin{cases} 3x + 3y = 60 \\ 3x + 5y = 90 \end{cases}$$

and then subtracting gives $2y = 30$, so $y = 15$.

And substituting $y = 15$ into $x + y = 20$ gives $x + 15 = 20$, so $x = 5$.

Our final intersection point is thus $(5, 15)$.

A general system-solving principle

If we have *any* system

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

we can multiply the top equation by d and the bottom by a to get:

$$\begin{cases} adx + bdy = cd \\ adx + aey = af \end{cases}$$

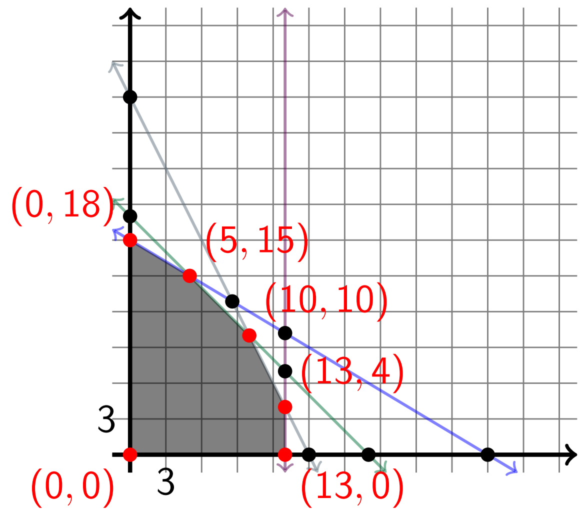
and subtract the two lines to get $bdy - aey = cd - af$.
Factoring and dividing gives $y = \frac{cd-af}{bd-ae}$.
If we substitute back into the top equation, then

$$ax + b\frac{cd-af}{bd-ae} = c$$

which can be solved to give $x = \frac{bf-ce}{bd-ae}$.
These are useful *methods*, but probably not worth applying as formulas.

Our feasibility region, fully labeled

$$\begin{cases} 2x + y \leq 30 \\ x + y \leq 20 \\ 3x + 5y \leq 90 \\ 2x \leq 26 \end{cases}$$



Note that the shaded region indicates which intersections are important: several others are infeasible and not worth calculating!

Finally maximizing profit

Aren't we done with this yet? (3.2.26)

Maximize $5x + 6y$ subject to $x, y \geq 0$ and $\begin{cases} 2x + y \leq 30 \\ x + y \leq 20 \\ 3x + 5y \leq 90 \\ 2x \leq 26 \end{cases}$

We test all corners $(0, 0)$, $(0, 18)$, $(5, 15)$, $(10, 10)$, $(13, 4)$, and $(13, 0)$.

$$5 \times 0 + 6 \times 0 = 0$$
$$5 \times 0 + 6 \times 18 = 108$$
$$5 \times 5 + 6 \times 15 = 115 \leftarrow \text{Maximum value 115 at } (5, 15)$$
$$5 \times 10 + 6 \times 10 = 110$$
$$5 \times 13 + 6 \times 4 = 89$$
$$5 \times 13 + 6 \times 0 = 65$$