Section 3.2: Putting it all together

MATH 105: Contemporary Mathematics

University of Louisville

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Everything in one place

Question 3.1.12/3.2.32

MPP makes downhill and cross-country skis. Downhill skis use 2 hours of cutting, 1 hour of shaping, and 3 hours of finishing; cross-country skis use 2 hours of cutting, 2 hours of shaping, and 1 hour of finishing. Every day the labor pool provides 140 worker-hours of cutting, 120 of shaping, and 150 of finishing. If downhill and cross-country skis yield respective profits of \$10 and \$8, what is the best production plan?

We approach this question in four stages:

- Conversion to mathematical abstraction
- Geometric representation of the feasible region
- Identification of feasible-region corners
- Testing objective function at each corner

Abstraction

Question 3.1.12/3.2.32

MPP makes downhill, x-country skis. Downhills use 2 hours, 1 hour, and 3 hours of cutting, shaping, and finishing respectively; x-countries use 2, 2, and 1 hour of the same. We have 140 cutting hours, 120 shaping, and 150 finishing. Downhill and x-country skis yield respective profits of $10 and $8.

Variables: x and y are # of downhill & x-country skis respectively.

Objective: We wish to maximize $10x + 8y$.

Constraints:
$$\begin{cases} 2x + 2y \leq 140 & \text{(Cutting)} \\ x + 2y \leq 120 & \text{(Shaping)} \\ 3x + y \leq 150 & \text{(Finishing)} \\ x, y \geq 0 \end{cases}$$

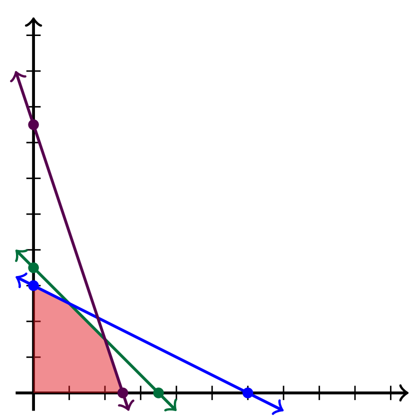
Visualization

Resource constraints for 3.1.12/3.2.32

$$\begin{cases} 2x + 2y \leq 140 & \text{(Cutting)} \\ x + 2y \leq 120 & \text{(Shaping)} \\ 3x + y \leq 150 & \text{(Finishing)} \end{cases}$$

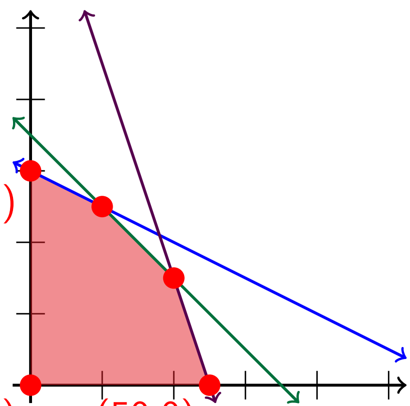
Intercepts:

- $\left(\frac{140}{2}, 0\right) = (70, 0)$
- $\left(0, \frac{140}{2}\right) = (0, 70)$
- $\left(\frac{120}{1}, 0\right) = (120, 0)$
- $\left(0, \frac{120}{2}\right) = (0, 60)$
- $\left(\frac{150}{3}, 0\right) = (50, 0)$
- $\left(0, \frac{150}{1}\right) = (0, 150)$



Coordinatization (visual)

$$\begin{cases} 2x + 2y \leq 140 \\ x + 2y \leq 120 \\ 3x + y \leq 150 \end{cases}$$



Those final coordinates are the solutions to the two systems of equations:

$$\begin{cases} 2x + 2y = 140 \\ x + 2y = 120 \end{cases} \text{ and } \begin{cases} 2x + 2y = 140 \\ 3x + y = 150 \end{cases}$$

Coordinatization (solving systems)

$$\begin{cases} 2x + 2y = 140 \\ x + 2y = 120 \end{cases}$$

Subtracting the two lines gives $(2x + 2y) - (x + 2y) = 140 - 120$, or $x = 20$.
Then $20 + 2y = 120$, so $2y = 100$, giving $y = 50$. One coordinate is $(20, 50)$.

$$\begin{cases} 2x + 2y = 140 \\ 3x + y = 150 \end{cases}$$

Doubling the second equation gives $6x + 2y = 300$; subtracting the first gives $(6x + 2y) - (2x + 2y) = 300 - 140$, so $4x = 160$ and thus $x = 40$.
Then $80 + 2y = 140$, so $2y = 60$, and thus $y = 30$. Our other coordinate is $(40, 30)$.

Maximization

Potential maximizing points are ~~$(0,0)$~~, $(50,0)$, $(40,30)$, $(20,50)$ and $(0,60)$.

Let's test each with our objective function $10x + 8y$:

$$\begin{aligned} 10 \times 50 + 8 \times \ \ 0 &= 500 \\ 10 \times 40 + 8 \times 30 &= 640 \\ 10 \times 20 + 8 \times 50 &= 600 \\ 10 \times \ \ 0 + 8 \times 60 &= 480 \end{aligned}$$

So the optimum strategy is to make **40 downhill and 30 cross-country skis** each day, for a profit of **\$640**.

Three-variable systems

When there are more than two products, there are more than two variables.

This follows the same basic theory, but is harder to visualize.

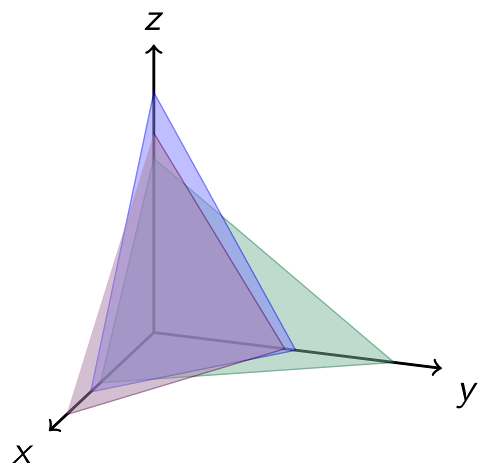
For purposes of illustration, we will look at, but not deal closely with, a linear programming problem in three variables.

There are a number of online services to solve these, including Wolfram Alpha.

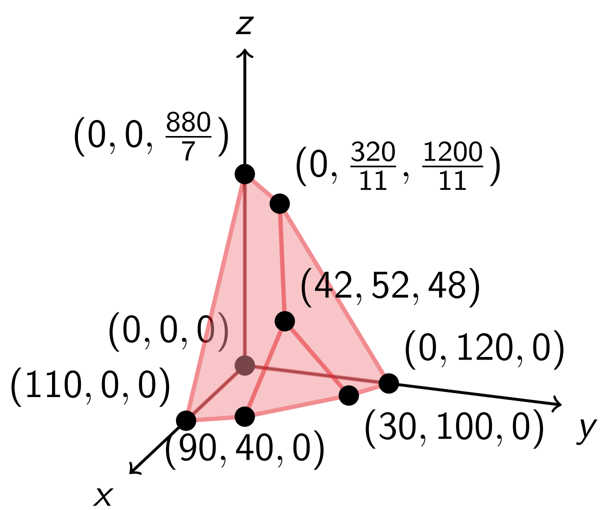
A visual example

Question 3.3.20

Maximize $8x + 6y + 7z$ for $x, y, z \geq 0$ & $\begin{cases} 0.4x + 0.2y + 0.35z \leq 44 \\ 0.2x + 0.2y + 0.15z \leq 26 \\ 0.4x + 0.6y + 0.50z \leq 72 \end{cases}$.



Feasible regions visualized



This weird shape has 6 faces and 8 vertices. Finding them is, in general, quite difficult!
But, once found, all of them can be tested. As it turns out, $8x + 6y + 7z$ is maximized at $(42, 52, 48)$, with a value of 984.