Pairwise comparison, and other methods

MATH 105: Contemporary Mathematics

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October 26, 2017

How to Simplify Voting

Every method we've seen so far seems flawed somehow. And yet choosing among alternatives *should* be easy!

One case where choosing among alternatives *is* easy is when there are only two options.

So one thing we might do is consider each voter's preferences as indicating how they *would* vote in an eection with only two candidates.

An example of a pairwise analysis

Here's an example similar to one we'd seen before, with 35 voters:

Number of votes	10	8	3	4	7	3
First choice	A	B	D	D	C	C
Second choice	B	A	C	B	B	D
Third choice	C	D	B	C	D	B
Fourth choice	D	C	A	A	A	A

We might look at how each head-to-head matchup would work out with these preferences:

vs.	D	C	B
A	18–17 (A)	18–17 (A)	10–25 (B)
B	25–10 (B)	22–13 (B)	
C	20–15 (C)		

So, D looks unusually awful, and B unusually good!

Vocabulary for interpreting the head-to-head matches

Any candidate who wins *every* pairwise comparison is called the *Condorcet candidate* or *Condorcet winner*.

Any candidate who loses *every* pairwise comparison is called the *Condocet loser*.

Any election may have both, one, or neither of these present.

Since a Condorcet candidate seems to be the "best" in some sense, it is desirable that the Condorcet candidate win:

Condorcet Fairness Criterion

A voting system satisfies the *Condorcet fairness criterion* if any candidate who wins every head-to-head match against every other candidate is guaranteed to win.

The Marquis de Condorcet was a contemporary of voting theory's other French Revlutionary thinker, Jean-Charles de Borda.

More about Condorcet fairness

None of the methods we've seen so far are Condorcet fair! Let's consider this schedule:

Number of votes	13	10	8	5	1
First choice	A	C	D	B	C
Second choice	B	B	C	D	D
Third choice	C	D	B	C	B
Fourth choice	D	A	A	A	A

Here, A would win the plurality vote with 13 votes.

B wins the Borda count with 107 points.

In IRV, B is eliminated with only 5 votes, and then C is eliminated with only 11, leaving D with a majority.

But *none* of these methods selected the Condorcet candidate, which was C (24–13 vs. A, 19–18 vs. B, 24–13 vs. A)!

Achieving Condorcet fairness

One way to achieve Condorcet fairness is to build a system *using* the pairwise comparisons to determine victory.
For instance, in the schedule given above:

Number of votes	13	10	8	5	1
First choice	A	C	D	B	C
Second choice	B	B	C	D	D
Third choice	C	D	B	C	B
Fourth choice	D	A	A	A	A

We have the following 6 pairwise matches:

vs.	D	C	B
A	13–24 (D)	13–24 (C)	13–24 (B)
B	28– 9 (B)	18–19 (C)	
C	24–13 (C)		

And we can use those to deterine "scores".

Achieving Condorcet fairness, continued

vs.	D	C	B
A	13–24 (D)	13–24 (C)	13–24 (B)
B	28– 9 (B)	18–19 (C)	
C	24–13 (C)		

We could consider every head-to-head match, awarding one point for victory and half a point for a tie, and then choosing the candidate with the highest score.

So C gets three points, B two, D one, and A none; C wins!

This calculation, knows as the *Method of Pairwise Comparisons* or *Copeland's Method*, does satisfy the Condorcet Fairness Criterion.

Like Borda count, Copeland's method was proposed in the 13th century by Ramon Llull.

Asessing Copeland's method

In many ways, Copeland's method seems to be ideal! It lacks the obvious problems of other methods.

- A majority candidate is guaranteed to win (majority candidates win every head-to-head match).
- Monotonicity fairness is achieved (improving a candidate improves their performance in the one-on-one matches).
- Third parties are recognized with minimal "spoiler" effects.
- Condorcet candidates are, by definition, guaranteed to win.

However, there are a number of problems with the method as well.

- Like Borda count and IRV, it requires complicated ballots.
- It requires a lot of tedious comparisons, especially in a large election.
- It is not responsive to small changes in the electorate.
- Because its point-scale is so granular, ties are very common.

Other methods

These four methods, and our three fairness criteria, will for the core of the necessary skills in this class.

However, there are many more voting systems, and ways to assess them, for the delight and interest of those curious about them.

One simple system which has gained interest is *approval voting*: it works just like plurality, but each voter can vote for multiple distinct candidates!

This system reduces the "spoiler" effect of third parties while still retaining the fundamental features of familiar plurality voting.

Even more other methods

Anti-plurality	Whoever gets the fewest last-place votes wins.
Coombs' method	Whoever gets the most last place votes is eliminated; repeat until a majority candidate emerges.
Nanson's method	Perform a Borda count; eliminate the lowest score and redo the Borda count with a smaller slate, repeating until one candidate remains.
Contingent voting	All except the two most popular candidates by plurality count are eliminated from the ballot, and then those two candidates are compared.

Other fairness criteria

- **Anti-majority** Whoever gets the most last-place votes shouldn't win.
- **Anti-Condorcet** The Condorcet loser shouldn't win.
- **Reversal Symmetry** If every preference order is reversed, the winner should change.
- **Consistency** If two sets of ballots, each won by the same candidate, are put together, then the resulting election should also be won by that candidate.

Just in case you thought Copeland's method was best: it violates consistency!