

# The search for a perfect voting system

MATH 105: Contemporary Mathematics

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## Review of Fairness Criteria

We've seen three fairness criteria so far:

**Majority** A majority candidate (when present) must win.

**Monotonicity** If a candidate is winning and then their position improves on some ballots, they should win the new election too.

**Condorcet** A Condorcet candidate (when present) must win.

Here's one more, and a very restrictive one:

### Independence of Irrelevant Alternatives

A voting method is *independent of irrelevant alternatives* if the winning candidate would remain the same after a losing candidate is removed from the ballot.

## Fairness of Plurality Voting

Plurality voting is majority-fair. If a candidate has a majority of votes, they also have a plurality.

Plurality voting is monotonic. If a candidate becomes more preferred, that can only increase their number of first-place votes, and cannot lose them the election.

Thus far, plurality looks pretty good, but it's not so great on the other criteria!

## Fairness of Plurality Voting (continued)

Consider this preference schedule with 3 candidates and 7 voters:

# of voters	3	2	2
1st choice	A	B	C
2nd choice	B	C	B
3rd choice	C	A	A

B is a Condorcet candidate (4–3 against A, 5–2 against C), but A wins a plurality vote. Thus plurality vote is not Condorcet-fair!

In addition, the removal of either of the losing candidates would change the winner from A to the remaining candidate, so plurality vote does not respect independence of irrelevant alternatives.

## Fairness of Borda Count

Borda count tends to look pretty bad according to fairness criteria. Despite its flaws, it *is* monotonic: an improvement in ballot position can only improve total Borda-count score.

Here's a simple 5-voter, 3-candidate example:

# of voters	3	2
1st choice	A	B
2nd choice	B	C
3rd choice	C	A

The Borda count yields scores of A: 11, B: 12, C: 7; B wins. But A is a majority (and thus a Condorcet) candidate, so this example demonstrates that Borda count is neither majority-fair nor Condorcet-fair.

It's not independent of irrelevant alternatives either, since removing losing candidate C will give victory to A.

## Fairness of IRV

Instant Runoff Voting is reasonably well-loved, but it has its problems too.

It is majority-fair, since any candidate with a majority to start with will win.

However, as we have seen before, it's not monotonic, and it won't satisfy the other criteria either.

## Fairness of IRV, continued

Here's a simple non-monotonicity example with 21 voters:

# of voters	8	5	2	6
1st choice	A	C	C	B
2nd choice	B	B	A	A
3rd choice	C	A	B	C

Here B would be eliminated, giving A 14 votes and victory.

If, however, the third cohort changed their votes to  $A > C > B$ , then C would be eliminated, giving B 11 votes and victory.

Thus IRV isn't monotonic, since improving A loses A the election.

Note that this example also shows that IRV isn't independent of irrelevant alternatives; removing losing candidate C would give victory to B.

## Fairness of IRV, continued

A variant of the example we gave for plurality vote will also demonstrate Condorcet-unfairness:

# of voters	4	2	3
1st choice	A	B	C
2nd choice	B	C	B
3rd choice	C	A	A

Here B is a Condorcet candidate (5–4 against A, 7–2 against C), but in IRV, B is eliminated to give victory to C with 5 votes.

## Fairness of Copeland's method

Copeland's method, fussy though it is, scores well on the standard fairness criteria.

By design, it selects Condorcet candidates, and thus will also select majority candidates.

It is also monotonic, since improving any candidate's ranking on ballots will improve their performance in head-to-head matches.

However, it is (perhaps unsurprisingly) not independent of irrelevant alternatives.

## Fairness of Copeland's method, continued

This ballot needs four candidates to break.

# of voters	1	1	2	2
1st choice	A	A	B	C
2nd choice	B	C	D	D
3rd choice	C	B	A	A
4th choice	D	D	C	B

Here A beats B (4–2), A beats C (4–2), and D beats A (4–2), while B ties C (3–3), B beats D (4–2), and C beats D (4–2). A wins with 2 points to B and C's 1.5 each and D's 0.

Drop B and C, however, and A is now in a head-to-head match with D, which D wins.

## Summary of fairness criteria

The four methods and four fairness criteria can be summarized as follows:

	Majority fair?	Monotonic?	Condorcet fair?	IIA?
Plurality voting	Yes	Yes	No	No
Borda count	No	Yes	No	No
Instant runoff voting	Yes	No	No	No
Copeland's method	Yes	Yes	Yes	No

IIA is often criticized as an over-restrictive criterion; very few systems satisfy it, and they are all systems which treat individual voters differently.

We might wonder if any system can be reasonably democratic without being somehow broken.

## Why democracy is doomed to fail

The answer is “No!” as determined by Kenneth Arrow in 1951.

### Arrow's Impossibility Theorem (as in the text)

No vote-tabulating system can satisfy both IIA and Majority Fairness.

Arrow's original statement didn't even require the system to be a democratic “voting” system:

### Arrow's General Impossibility Theorem

No decision-making system choosing among three or more alternatives using preference listings satisfies all three of the following criteria:

- ▶ Removing nonwinning alternatives does not affect the outcome.
- ▶ If every person in the society wants an outcome, it is selected.
- ▶ The system does not simply mimic the choice of a single “dictator”.

## Why democracy is doomed to fail, continued

Not only is no voting system guaranteed to satisfy basic criteria of justice, but every voting system encourages dishonesty.

### Gibbard-Satterthwaite Theorem

In any decision-making system which is not a dictatorship using preference listings among at least three alternatives, there are circumstances where one voter achieves the best goal for themselves by voting in a way which does not reflect their actual preferences.

We saw this in Florida 2000, and in the non-monotonic IRV examples.

## Now what?

Democracy sucks. What can we do about it?

We can't fix voting outright, but some systems are less fragile than others.

A greater awareness of novel systems guarantees that, even if we can't get a voting system that always works perfectly, we can find one that aligns well with our values.