Section 4.4: The Basics of Apportionment

MATH 105: Contemporary Mathematics

University of Louisville

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What is apportionment?

Most democracies are not *direct* democracies, where the electorate at large is polled for every decision.

In a large society, with complicated questions, and an electorate with other concerns, it would be impractical to share information, debate, and decide among every single person.

So most of our democratic systems work via *representation*, where certain (usually geographical) cohorts each get to select someone to represent them in the process of governance.

Apportionment is the process by which a number of representatives is assigned to each cohort.

In addition, the process of *districting* is used to assign a geographic region to each representative.

Contexts for apportionment and districting

In some nations with parliamentary systems, voters select not a candidate by a party, and then seats in the parliament are *apportioned* by number of votes.

In America, apportionment is done on both the national and state level by geography.

- Each state is apportioned a number of representatives to Congress.
- Within state legislatures, representatives are often doled out county-by-county.

The subordinate process of *districting* is usually done by the states.Districting is a much hotter topic today than apportionment, but both can be complicated and surprisingly difficult to do fairly.

Legal basis for apportionment

Article I, Section 2 of the US Constitution

Representatives and direct Taxes shall be apportioned among the several States which may be included within this Union, according to their respective Numbers... The Number of Representatives shall not exceed one for every thirty Thousand, but each State shall have at Least one Representative.

This gives very little guidance; it says the number of representatives should be proportional to the population, and that there should be no more than $\frac{1}{30000}$ times the population, but little else!
Some case law has clarified the situation somewhat; e.g. the 1964 case *Wesberry v. Sanders* established that each representative's district should enclose nearly the same population.
However, historically, the potential for error and unusual interpretation has been fertile ground for political scheming.

Why should apportionment be hard?

Districting is a horrible can of worms; but why is apportionment troublesome?

Sometimes it's not; suppose we wanted to assign a 20-person proportional legislative body to the following four counties in a state with 800,000 people:

County	Population
Smith	320,000
Burton	160,000
Clark	120,000
Ocean	200,000

20 representatives, split evenly over 800,000 people, should assign one representative to each group of 40,000 people.

Logically, then, Clark County gets 3, Burton 4, Ocean 5, and Smith 8.

Generalizing our process

In the last slide, we saw that a 20-person representative body for 800,000 people should have one representative per 40,000 people. Why?

If we have n representatives, and a population P, then we want one representative per $\frac{P}{n}$ people.

This number is known as the *average constituency size* or the *standard divisor*.

Once we have the standard divisor, we *divide* each region's population by it to get their *standard quota* of representatives.

So apportionment should be really easy!

So, why not give every region its standard quota of representatives?

Well, let's suppose that the four counties measured above have unequal growth, and ten years later, have different populations:

County	Population	Quota of reps
Smith	350,000	7.37
Burton	190,000	4.00
Clark	150,000	3.16
Ocean	260,000	5.47

Our total population has grown to 950,000, for a standard divisor of 47500.

So to find each county's quota, we divide its population by 47500 above.

How can we possibly select these fractional numbers of people?

Interlude: An American history lesson

The first census was taken in 1790; at that time, the House of Representatives had 105 members.

Having to "round off" non-integer quotas became a problem almost immediately in American history.

Over the course of the next 150 years, America would use four different approaches to rounding off these non-integer parts.

Other nations also confronted these problems, and their own solutions incuded both the same ones used in America and other approaches.

A simple and crude approach

Recall our original plan for apportioning 20 representatives:

County	Population	Quota of reps
Smith	350,000	7.37
Burton	190,000	4.00
Clark	150,000	3.16
Ocean	260,000	5.47

If we round *down*, then we will assign 7 delegates to Smith County, 4 to Burton, 3 to Clark, 5 to Ocean.

But this only doles out 19 reps, and we ought to have one more.

The most badly cheated on their standard quota by this scheme would be Ocean County, so we'll give them the extra delegate in recompense.

Formalizing what we just did

So, the process we followed on the last slide can be generalized. Suppose we have a bunch of regions and a number of representatives N. Here's how we might apportion them:

- Divide the total population by N to get the *standard divisor* D.
- Divide each region's population by D to get their individual *standard quotas*.
- Round down the standard quotas to get each state's *lower quota*.
- Add up the lower quotas to determine how many representatives are unassigned.
- Give out that many "bonus reps" one at a time to regions, starting with the region whose standard quota has the largest fractional part.

This approach is variously known as *Hamilton's method*, *Vinson's method*, the *Hare-Niemeyer method*, or the *Method of Largest Remainders*.

Another Hamilton Method example

Let's look at how a city council of 99 people might be apportioned among four neighborhoods:

N'hood	Pop.	Std. quota	Lower quota	Bonus	Total
Forkland	2060	10.1970	10		10
Knifeton	2080	10.2960	10	+1	11
Spoonville	7730	38.2635	38		38
Platesburg	8130	40.2435	40		40
Total	20000	99.0000	98		99

We calculate the standard divisor $D = \frac{20000}{99} \approx 202.02$.
Then we divide each state's population by this value D.
Now we round all the quotas down.
But, since we are one delegate short, we assign a bonus delegate to whichever state had the highest decimal part in its SQ.

Tweaking that example

Suppose they decide the 99-person city council is an ungainly number, and bump it up to a nice round 100?

N'hood	Pop.	Std. quota	Lower quota	Bonus	Total
Forkland	2060	10.30	10		10
Knifeton	2080	10.40	10		10
Spoonville	7730	38.65	38	+1	39
Platesburg	8130	40.90	40	+1	41
Total	20000	100.00	98		100

The standard divisor is now $\frac{20000}{100} = 200$.

A paradox, a paradox, a most ingenious paradox!

Let's summarize how the Hamilton apportionment proposed to split out either 99 or 100 representatives among those four neighborhoods:

Neighborhood	99-rep distribution	100-rep distribution
Forkland	10	10
Knifeton	11	10
Spoonville	38	39
Platesburg	40	41
Total	99	100

The good people of Knifeton have cause to feel cheated by the addition of a new council member!

What has happened here is a weakness of the Hamilton apportionment technique known as the *Alabama Paradox*.

Pair o' docs, pair o' delegates

The Alabama Paradox

An apportionment technique is subject to the *Alabama Paradox* when an increase in the total number of representatives, without any change in population, results in one of the regions *losing* representatives.

This paradox was discovered in 1880, when a detailed study of potential sizes for the House of Representatives determined that in a House with 299 members, Alabama would get 8, while with 300 members, it would only get 7.

Nonetheless, the US continued to use the Hamilton method from 1850 to 1900. In 1900 the Paradox hit Colorado, which was apportioned three seats for every House size between 350 and 400 *except* for 357, at which it would only have gotten 2.