1. **(20 points)** *Answer the following questions about counting strings, sets, and partitions.*

   (a) **(10 points)** *Recall that the number of strings of 5 distinct letters chosen from an alphabet of 7 is* $\frac{7!}{2!} = 2520$*. We divide this by* 120 *to get the number of sets of 5 distinct letters (*21*).*

   *We also know that there are* $7^5 = 16807$ *strings of 5 freely chosen letters chosen from an alphabet of 7. Why can't we use a similar procedure to the one above to get the number of multisets of 5 letters from these 7 (of which there are 462, for reference). Describe the failure of this method in terms of the objects being investigated; the fact that* $7^5$ *isn't divisible by 120 isn't enough justification!*

   The first calculation worked because every string of 5 *distinct* letters has exactly 120 rearrangements, so each set is associated with exactly 120 strings. There is no such convenient fact for multisets and strings of potentially identical letters: the multiset {A, A, A, A, A} is only assoicated with one string, AAAAA, while {A, A, A, B, B} has 10 associated strings (AAABB, AABAB, AABBA, ABAAB, ABABA, ABBAA, BAAAB, BAABA, BABAA, and BBAAA), and so forth. Because the multiset-to-string correspondence is not one-to-many with a consistent value of "many", there is no easy division to get from a count of the strings to a count of the multisets.

   (b) **(10 points)** *There are, as seen in class, 150 strings of length 5 using the letters A, B, and C, and using each letter at least once. There are 25 ways to partition the set* $\{1, 2, 3, 4, 5\}$ *into three nonempty subsets. Describe how these two things being counted are related, and, specifically, how* $150 = 6 \times 25$ *is relevant to the counting.*

   Here both scenarios we are considering can be interpreted as a five-balls-into-three-boxes paradigm with the restriction that each box is nonempty (this condition corresponds in the two cases respectively to the requirement that each letter appear once and the requirement that each of the sets are nonempty). The main difference, however, is that in the first scenario the boxes are labeled (the string "ABBAC" is different from "CAACB" because A, B, and C are distinguishable letters), whereas in the second they are not (the partition into the sets $\{1, 4\}$, $\{2, 3\}$, and $\{5\}$ is the same as the partition into the sets $\{2, 3\}$, $\{5\}$, and $\{1, 4\}$). However, if we were to consider how we can label the boxes, we shall see that every partition admits exactly six labelings: the boxes are already intrinsically distinguished by their contents, so there are 3 choices as to which box to label "A", and then, two choices of which remaining box to label "B", and then only one choice for where the label "C" goes. Thus each partiton can be transformed into six distinct strings: by way of illustration, we could observe that the above-mentioned partition into $\{1, 4\}$, $\{2, 3\}$, and $\{5\}$ determines specifically the 6 strings ABBAC, ACCAB, BAABC, BCCBA, CAACB, and CBBCA.

   Note that this division process would *not* quite work between the 243 free strings and the 41 partitions with possibly empty sets, because the partition into $\{1, 2, 3, 4, 5\}$ and two empty sets only has 3 associated strings, not 6. This occurs because there is no intrinsic "distinguishability" between the two empty sets.

2. **(10 points)** *In some alien variant of poker played with a standard deck of cards, a* periflush *is a hand of five cards where two cards have the same number, and the other three cards are all of the same suit as each other and one of those two cards: for instance,* 4♡ 4♣ *J*♡ 3♡ 7♡ *is a periflush since the fours form a pair, and the other three cards together with one element*

*of the pair are all of the same suit, while* $9\spadesuit\ 9\heartsuit\ 3\heartsuit\ Q\spadesuit\ A\heartsuit$ *is not, since the non-pair cards are not all of the same suit. What is the probability that, drawing five cards at random, you draw a periflush? Recall that a standard poker deck consists of 52 distinct cards, with 13 different numbers and 4 different suits.*

There are many different possible calculations, but they all should give the same answer when multiplied out. We might select any of the 13 possible values for the pair, and then any of the $\binom{4}{2}$ possibilities for the suits used in the pair. Then one of those two suits should be chosen as the "mini-flush" suit, so there is a choice from among 2 alternatives to be made there. Finally, the values for the remaining three cards in the mini-flush must be selected from among the 12 values not appearing in the pair, so there are $\binom{12}{3}$ possible choices there. Multiplying all these together, there are

$$13 \cdot \binom{4}{2} \cdot 2 \cdot \binom{12}{3} = 34320$$

possible periflushes. Since there are $\binom{52}{5} = 2598960$ possible hands in total, the probability of a periflush is $\frac{34320}{2598960} \approx 1.32\%$. For reference, this is less likely than drawing a three-of-a-kind (2.11%) and more likely than a straight (0.76%).

3. **(10 points)** *Prove* combinatorially *that, for any integers* $k$ *and* $n$ *with* $0 \leq k \leq n$,

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

One simple way to view both sides of this equation is as an expression of the number of ways to select a $k$-person committee which has a chairperson on that committee (or a team with a captain, or any other group-and-distinguished-member idiom) out of a pool of $n$ candidates. We might start by selecting a committee, which we do by choosing $k$ people from the $n$, in any of $\binom{n}{k}$ ways, and then follow up by appointing one of the chosen members of the committee to be the chair: since there are $k$ people on the committee, there are $k$ choices as to how to sleect a chair. Thus, there are $k\binom{n}{k}$ ways to build such a committee.

However, alternatively, we might build our committee by selecting a chair first, from our candidate pool. Since there are $n$ candidates, there are $n$ possible ways to make this decision. Having done so, we still have $k-1$ seats to fill on our committee, and $n-1$ remaining candidates who could fill them. We can thus select the non-chair members of our committee in $\binom{n-1}{k-1}$ ways, for a total of $n\binom{n-1}{k-1}$ ways to build the committee.

Since both $k\binom{n}{k}$ and $n\binom{n-1}{k-1}$ count the number of ways to build this structure, we see that these two quantities must be equal.

Musica est exercitium arithmeticae occultum nescientis se numerare animi. [Music is an arithmetical exercise of the soul, which is unaware that it is counting.]

—Gottfried Leibniz