Show all your work, and explain why you use the arithmetic operations you use in reaching an answer.

1. **(10 points)** Prove *combinatorially* that for any non-negative m and n,

$$\sum_{k=m}^{n} \binom{n}{k} \left\{ {k \atop m} \right\} = \left\{ {n+1 \atop m+1} \right\}.$$

2. **(10 points)** Recall that a permutation of the sequence of numbers $(1, 2, 3, \ldots, n)$ which does not have any i in the ith position is called a *derangement*, and that the number of derangements of length n is called D_n. Prove *combinatorially* that for $n \geq 2$,

$$D_n = (n-1)(D_{n-1} + D_{n-2}).$$

   Hint: consider separately the cases when 1 is simply swapped with a different number k, versus when it is moved to a position k while the value k goes somewhere else completely.

3. **(10 points)** Use inclusion-exclusion (i.e., not brute force) to determine the number of bitstrings (i.e. strings of the characters "0" and "1") of length 7 which do not contain the substring "1111".

4. **(10 points)** A jar contains 5 identical red balls, 6 identical blue balls, and far more identical green balls than you would ever want. Let a_n be the number of different n-element collections you could draw from this jar such that you draw at least 2 red balls, and the number of green balls drawn is a multiple of 3. For instance, $a_5 = 5$ because the triple of red, green, and blue ball counts could be any of $(2, 3, 0)$, $(2, 0, 3)$, $(3, 2, 0)$, $(4, 1, 0)$, or $(5, 0, 0)$. Answer the following questions about this system.

   (a) Write a rational function which represents the ordinary generating function $\sum_{n=0}^{\infty} a_n x^n$.

   (b) Calculate a_{10}.

The worst thing you can do is to completely solve a problem. —Dan Kleitman.