1. **(10 points)** Let $p_n(k)$ represent the number of partitions of the integer k into exactly n positive integer parts, as seen in class.

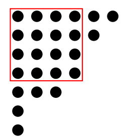
   (a) **(7 points)** Prove combinatorially that for any positive k and n,

   $$p_n(k) = p_{n-1}(k-1) + p_n(k-n)$$

   (Hint: a partition either has a part of size 1 or it does not.)

   (b) **(3 points)** Using the above recurrence, together with the fact that $p_0(0) = 1$ and $p_n(k) = 0$ when $k < n$ or $n = 0$ and $k > 0$, calculate $p_4(10)$.

2. **(10 points)** Let a partition be said to have a *squariness* equal to the side-length of the largest square which fits into the upper left corner of the Ferrers diagram. For instance, the partition of 24 into $6+5+4+4+3+1+1$, as depicted below, has a "squariness" of 4.



   (a) **(5 points)** Prove combinatorially that the number of partitions of the number n which have squariness of s is

   $$\sum_{i=0}^{n-s^2} p_s(i+s)p_s(n-s^2-i+s).$$

   (b) **(5 points)** Use the decomposition you produced in the previous part to argue that

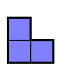
   $$\prod_{k=0}^{\infty} \frac{1}{1-x^k} = \sum_{s=0}^{\infty} x^{(s^2)} \left(\prod_{i=1}^{s} \frac{1}{1-x^i} \right)^2$$

3. **(10 points)** Let a_n represent the number of strings consisting of the letters A, B, C, and D with at least one C, at least two Ds, and with an even number of Bs.

   (a) **(6 points)** Produce a closed formula for the exponential generating function $\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$.

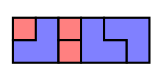
   (b) **(4 points)** Use the above generating function to calculate a_{10}.

4. **(10 points)** Let t_n represent the number of ways to completely tile a $n \times 2$ rectangle with any number of copies of the "L" triominoes depicted below (in any orientation) and 1×1 squares. A single example of a tiling of a 6×2 rectangle is shown:



"L" triomino

1×1 square



Tiling of a 6×2 rectangle

   (a) **(5 points)** Produce (with explanation) a recurrence relation and initial conditions for t_n.

   (b) **(5 points)** Use your recurrence to produce a closed formula for t_n.

A matematikus olyan gép, amely kávéból tételeket gyárt. —Alfréd Rényi Kleitman

 due October 31, 2017