1. **(10 points + 10 bonus points)** Let F_n represent the Fibonacci numbers as defined by $F_0 = 1$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Recall also that the Fibonacci nubmers enumerate the tilings of a $1 \times n$ rectangle with dominoes and checkers. Prove the following two identities combinatorially.

   (a) **(10 points)** For any $n \geq 0$, $f_{2n+1} = \sum_{i=0}^{n} f_{2i}$.

   (b) **(5 point bonus)** For any $n \geq 2$, $3f_n = f_{n+2} + f_{n-2}$.

   (c) **(5 point bonus)** For any $n \geq 0$, $f_n f_{n+1} = \sum_{k=0}^{n} f_k^2$.

2. **(10 points)** You have three colors of paint and want to paint the faces of a cube using every color at least once. If two colorings are considered to be identical if one could be rotated onto the other, how many ways are there to do this?

   (Hint: note that, including the "do-nothing" rotation, there are 24 ways to rotate a cube.)

3. **(10 points)** Either draw a graph with 8 vertices, all of odd degree, which does not contain a path of length 3, or explain why such a graph can't exist.

4. **(10 points)** A "greedy" coloring of a graph is one which looks at each vertex in turn, and chooses the smallest color which is not yet used on its neighbors. Show that for each n there is a 2-colorable graph on $2n$ vertices which, if greedily colored in the wrong order, needs n vertices. This is a way to show that greedy coloring can do a very bad job indeed!

> The best of ideas is hurt by uncritical acceptance and thrives on critical examination.
> —George Polya