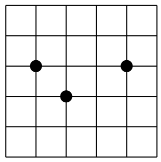
1. **(10 points)** *How many direct paths are there from the lower left corner to the upper right corner of the following grid which* pass through at least one of the marked points*?*



Let A, B, and C be the sets of paths through $(2,2)$, $(1,3)$, and $(4,3)$ respectively. We want to find $|A \cup B \cup C|$ here, and will use inclusion-exclusion. Note that $|A| = \binom{4}{2}\binom{6}{3} = 120$, $|B| = \binom{4}{1}\binom{6}{2} = 60$, and $|C| = \binom{7}{3}\binom{3}{1} = 105$. Looking at intersections, $|A \cap B| = 0$, since no path goes through both $(2,2)$ and $(1,3)$, while $|A \cap C| = \binom{4}{2}\binom{3}{1}\binom{3}{1} = 54$ and $|B \cap C| = \binom{4}{1}\binom{3}{0}\binom{3}{1} = 12$. Thus:

$$|A \cup B \cup C| = 120 + 60 + 105 - 0 - 54 - 12 + 0 = 219.$$

2. **(15 points)** *Show via a combinatorial proof that for* $0 \le k \le n$*,*

$$\sum_{j=k}^{n} \binom{n}{j}\binom{j}{k} = \binom{n}{k} 2^{n-k}.$$

The right side counts what is clearly a selection in two parts: a "special subset" of k elements of $\{1, 2, 3, \ldots, n\}$, together with a subset built from any number of the remaining $n-k$ elements; that is to say, on the right side we construct a pair of *disjoint* subsets (A, B) of $\{1, \ldots, n\}$, with $|A| = k$. On the left side, however, we construct a set of at least k elements (specifically, of j elements, with j ranging from k to n), and then construct a subset of exactly k elements therefrom. So on the left side we're building pairs (R, S) where $R \supseteq S$, and $|S| = k$. We can easily relate these two constructions to each other by letting $A = S$ and $R = A \cup B$, or, alternatively, letting $B = R - S$.

A more folksy explanation might be that this describes the number of ways of recruiting for an organization with a k-person board of directors but otherwise of no fixed size from a population of n people. The left and right sides describe two different processes for achieving this result: the three symbols on the left represent, in order, deciding how large the organization will be (denoting it j, and permitting any size from k on up), which members of the population will join (there are $\binom{n}{j}$ possibilities), and then, from among the just-selected members of the organization, further selecting k of them to be on the board of directors (in any of $\binom{j}{k}$ ways); on the right, we instead first select the board from the general population (in any of $\binom{n}{k}$ ways) and, for each of the remaining $n-k$ members of the population, make a binary choice as to whether they are in the organization or not.

3. **(5 points)** *What is the coefficient of* x^2 *in the expansion of* $(3x+4)^6$*?*

We know that the binomial expansion yields the term $\binom{6}{2}(3x)^2(4)^4$, so segregating out the constants yields $3^2 4^4 \binom{6}{2} x^2 = 34560x^2$ so the coefficient is 34560.

4. **(25 points)** *A far-future dystopia is replacing everyone's names with 7-digit codes using only the digits 1, 2, 3, and 4.*

(a) **(5 points)** *How many different "names" are possible under this scheme?*

Each digit of someone's name could be any of 4 different values, and a name is made up of 7 such choices, so there are $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^7 = 16384$ names.

(b) **(10 points)** *Two people are said to be in the same artificial family if their names are anagrams of each other; for instance, citizens 1314221 and 2211134 are both in the same family. How many different families are there?*

A "family" here is just a set of digit-frequencies; for instance, the above family is a three-ones, two-twos, single-three-and-four family. Thus, a family can be represented by the distribution of seven *unlabeled* balls to four labeled boxes, which is best modeled with a balls-and-walls paradigm of seven balls and three walls, which can be distributed in $\binom{10}{3} = 105$ ways for a total of 105 families.

Note that families come in widely varying sizes. Four families (the "all-1s", "all-2s", "all-3s", and "all-4s") have a single member each, while four families (each with a 2+2+2+1 distribution of name digits) would have $\binom{7}{2,2,2,1} = 630$ members. The *average* family size would be $\frac{4^7}{\binom{10}{3}} \approx 156.04$, which is of course not an integer but there is no good reason to expect an average to be an integer.

5. **(10 points)** *Describe (with brief justification) in big-O notation the time taken by the following procedure for finding the dot products of two n-dimensional vectors* $\mathbf{a}$ *and* $\mathbf{b}$.

- *Initialize* $z \leftarrow 0$.
- *For each value of i from 1 to n:*
  - *Assign* $z \leftarrow z + a_i b_i$.
- *Return the value of z and terminate.*

Most of the individual steps take constant time, however the single step "Assign $z \leftarrow z + a_i b_i$" is performed n different times, for n different values of i. Because of this the overall run time is linear in n, described as $O(n)$.

6. **(15 points)** *Galangal is a ginger-like rhizome used in Southeast Asian cuisine. Answer the following questions about anagrams of the word "GALANGAL".*

(a) **(5 points)** *How many anagrams are there in total?*

There are 2 Ls, 2 Gs, 3 Ls, and a single N, so a multinomial coefficient is an effective measure: $\binom{8}{2,2,3,1} = \frac{8!}{2!2!3!1!} = 1680$.

(b) **(10 points)** *How many anagrams are there which do not contain either a double G or a double L (i.e. "GLAANLGA" is OK, "LANGGLAA" is not).*

From the above count of anagrams, we wish to subtract off those which use a double G or a double L; in addition, we will want to add back in those which use both, for inclusion-exclusion purposes.

To count these forbiddences, we could consider the possibility of "gluing" the Gs or Ls together. So now we want anagrams of two Gs, three As, an N, and an LL, which can be made in any of $\binom{7}{2,3,1,1} = 420$ ways. In total, we will end up with:

$$\binom{8}{3,2,2,1} - \binom{7}{3,2,1,1} - \binom{7}{3,2,1,1} + \binom{6}{3,1,1,1} = 960.$$