For each question, explain your reasoning in full sentences.

1. *For two sets A and B, suppose that all you know about the two of them is that $|A| = 6$ and $|B| = 4$.*

   (a) *What is the largest possible value of $|A \cup B|$? What is its smallest possible value?*

   Since elements which appear in both sets would only appear once in $A \cup B$, and since $A \cup B$ contains those elements which are in either A or B, this set's size is minimized by letting A and B overlap as much as possible, e.g. $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 2, 3, 4\}$. Under such a circumstance, we can see that $|A \cup B| = 6$.

   Likewise, in order to maximize the union, we would want to make sure that each element of A and each element of B contributes to the number of elements in $A \cup B$; this is achieved by minimizing the overlap. So, for instance, if $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{7, 8, 9, 10\}$, then $|A \cup B| = 10$.

   (b) *What is the largest possible value of $|A \cap B|$? What is its smallest possible value?*

   Since $A \cap B$ consists of those elements which appear in both sets, it is maximized when the overlap between the two sets is maximized, as in the example above where $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 2, 3, 4\}$; under this circumstance $|A \cap B| = 4$.

   Two sets of arbitrasry size can certainly be disjoint, and if two sets are disjoint then their intersection is the set of smallest possible size, to wit, the empty set, with 0 elements. As in the previous question, we might look at the specific example of $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{7, 8, 9, 10\}$ to get $|A \cap B| = 0$.

   (c) *What is the largest possible value of $|A - B|$? What is its smallest possible value?*

   If B is disjoint from A, then $A - B$ will simply be the same set as A, since removing the elements of B removes nothing at all. This is going to be the largest that $A - B$ can be, so $|A - B| = 6$ in this case. If, however, B is a subset of A, then $A - B$ will lack four elements of A, so $|A - B|$ will be 2, as small as possible.

   (d) *What is the largest possible value of $|B - A|$? What is its smallest possible value?*

   As above, if A and B are disjoint, then $B - A$ will simply be the same set as B, since removing the elements of A removes nothing at all. This is going to be the largest that $B - A$ can be, so $|B - A| = 4$ in this case. If, however, B is a subset of A, then $B - A$ will be empty, since every element of B is an element of A and is thus excised from $B - A$. Thus $|B - A| = 0$, the smallest it (or any set's cardinality) can be.

2. *Describe two sets S and T such that S is both an element* and *a subset of T.*

   There are many possible answers. The most absurdly simple is $S = \emptyset$ and $T = \{\emptyset\}$; then $S \subseteq T$ since the empty set is a subset of every set, and $S \in T$ because T's sole element is in fact the empty set. A more conventional answer would be something akin to letting $S = \{1, 2\}$ and $T = \big\{1, 2, \{1, 2\}\big\}$.

3. *If X is an infinite set and $X \in Y$, can you be certain that Y is also infinite?*

   No! Since X is an *element* of Y, it contributes exactly one to the number of elements in Y, which might well be finite. Indeed, as a simple example we might consider when X is the infinite set $\mathbb{N}$, and Y is the one-element (and thus finite) set $\{\mathbb{N}\}$.

4. *If X is an infinite set and $X \subseteq Y$, can you be certain that Y is also infinite?*

   This is very different from the case above! Subset inclusion forces the superset Y's elements to include every element of its subset X. If X has more elements than can be enumerated with a finite value, all of these elements are then elements of Y, causing Y to be an infinite set.

5. *Does it appear to be possible or meaningful to you for there to be a set S which is an element of itself, that is, a set S such that $S \in S$? Does this appear to be either a useful or problematic concept? Your response to this need not be "mathematically justified", but should express a personal consideration of the ramifications of such an idea.*

   This is not an issue on which set theory constructions are unified. Historically, the notion of "a set" has simply been "a test for membersip, applicable to every object conceivable", which would allow a set to contain itself — just let it answer "yes" when tested for membership of itself! But this simplistic notion of what kind of set can be defined is fraught with peril: Bertrand Russell proposed the definition of a set S whose criterion for membership is that it contains all sets which do not contain themselves. This definition, known as *Russell's paradox*, is a variation on an older puzzle known as the *barber paradox*, and ends with the self-contradictory notion that S contains itself if and only if it does not contain itself!

   The solution to Russell's paradox has required a much more rigorous definition of how sets can be defined, and a constraint on the extent to which their definitions can refer to themselves. However, depending on the underlying assumptions of the set theory, self-containment can be either forbidden outright or allowed in a heavily controlled form. The most conventional set-theoretical systems used today are the Zermelo-Fraenkel (ZF) and Zermelo-Fraenkel-Choice (ZFC) axiom sets, which specifically forbid self-containment. Quine's New Foundations (NF) system, among other unconventional set theories, does allow self-containment of sets, but uses additional restrictions to avoid falling afoul of Russell's papradox.

   Fortunately, the question of whether self-containment is possible will have no impact on our ability to do anything we need ot do in this course, but for my own part, I find that self-containment is an insufficiently useful concept to justify the paradoxical behavior which seems to be a near-inevitable byproduct of allowing it.