For each question, explain your reasoning in full sentences.

1. *For each of the following conditions, either find a partition of $\mathbb{N}$ satisfying that condition, or explain why such a thing is impossible.*

   (a) *A finite partition $\mathcal{A}$ such that each element of $\mathcal{A}$ is an infinite set.*

   There are many ways to do this, but the most straightforward way to partition $\mathbb{N}$ into a finite number of parts is to bisect it along the lines of *parity*, putting all even nubmers in one part and all odd numbers in the other. This could be written as

   $$\mathcal{A} = \big\{\{1, 3, 5, 7, \ldots\}, \{2, 4, 6, 8, \ldots\}\big\},$$

   or more explicitly as

   $$\mathcal{A} = \big\{\{2n : n \in \mathbb{N}\}, \{2n - 1 : n \in \mathbb{N}\}\big\}.$$

   Note that $|\mathcal{A}| = 2$, and that each of the two elements of $\mathcal{A}$ are themselves infinite sets; these two sets indeed are disjoint and have $\mathbb{N}$ as their union, verifying that $\mathcal{A}$ is a partition. An even simpler (and arguably trivial) choice would be to let $\mathcal{A} = \{\mathbb{N}\}$.

   (b) *An infinite partition $\mathcal{B}$ such that each element of $\mathcal{B}$ is an infinite set.*

   This is fairly difficult, although there are a great many ways to do it. The most straightforward approach is to further ramify the classification by parity presented above. Above we divided the numbers into those which are divisible by 2 and those which are not. We could further carve out of the first class those which are and are not divisible by 4; then we could further split this divisible-by-4 group into those individisble and divisible by 8, and so forth. This particular approach will yield the partition

   $$\mathcal{B} = \big\{\{1, 3, 5, 7, \ldots\}, \{2, 6, 10, 14, \ldots\}, \\ \{4, 12, 20, 28, \ldots\}, \{8, 24, 40, 56, \ldots\}, \ldots\big\},$$

   which could also be written more explicitly as

   $$\mathcal{B} = \{B_i : i \in \mathbb{N}\} \text{ where } B_i = \{2^i(2n - 1) : n \in \mathbb{N}\}.$$

   (c) *A finite partition $\mathcal{C}$ such that $\mathcal{C}$ has both infinite and finite sets as elements.*

   The easiest way to do this would be to carve a finite set out of one of the parts of $\mathcal{A}$ above. For instance, one could pull a finite number of odd elements out of the pdd part to get

   $$\mathcal{C} = \big\{\{1, 3\}, \{5, 7, 9, 11, \ldots\}, \{2, 4, 6, 8, \ldots\}\big\}$$

   or, even more simply, to make a collection with some finite set S of natural numbers and the infinite set $\mathbb{N} - S$ as elements, e.g., with $S = \{1\}$, this would be

   $$\mathcal{C} = \big\{\{1\}, \mathbb{N} - \{1\}\big\}\big\{\{1\}, \{2, 3, 4, 5, 6, \ldots\}\big\}.$$

   (d) *An infinite partition $\mathcal{D}$ such that $\mathcal{D}$ has both infinite and finite sets as elements.*

   One way to do this would be to take a partition with two or more infinite parts, such as the partition by parity, and ramify one of the infinite parts into an infinite quantity of single-element parts. For instance, we might put every odd number in the same part, and give each even number its own part to have the partition

   $$\mathcal{D} = \big\{\{1, 3, 5, 7, 9, \ldots\}, \{2\}, \{4\}, \{6\}, \{8\}, \{10\}, \ldots\big\}$$

   If we wanted to write this more explicitly, without ellipses, we could say

   $$\mathcal{D} = \{D_i : i \in \mathbb{N}\} \text{ where } D_1 = \{2n - 1 : n \in \mathbb{N}\} \text{ and } D_i = \{2i - 2\} \text{ for each } i > 1.$$

(e) *A finite partition $\mathcal{E}$ such that each element of $\mathcal{E}$ is a finite set.*

This is impossible. Suppose that $\mathcal{E} = \{E_1, E_2, \ldots, E_n\}$ where each E_i is a finite nonempty set. Thus each E_i has a greatest element e_i; taking the maximum of this finite list of e_is, we get an element N which is the largest appearing in any E_i. Then $N+1$ isn't in any E_i, so $N+1 \notin \bigcup_{i=1}^n E_i = \bigcup_{E\in\mathcal{E}} E$. But in order for $\mathcal{E}$ to be a partition of $\mathbb{N}$, every element of $\mathbb{N}$ would have to be an element of $\bigcup_{E\in\mathcal{E}} E$.

(f) *An infinite partition $\mathcal{F}$ such that each element of $\mathcal{F}$ is a finite set.*

One straightforward possibility is to partition every element of $\mathbb{N}$ into its own single-element part, for instance by letting

$$\mathcal{F} = \big\{\{1\}, \{2\}, \{3\}, \{4\}, \ldots\big\},$$

or more explicitly as

$$\mathcal{F} = \big\{\{i\} : i \in \mathbb{N}\big\}.$$

2. *Explain (in whatever words appear to constitute a convincing argument) why it should generally be the case that $A \times (B \cup C) = (A \times B) \cup (A \times C)$ for sets A, B, and C.*

Let's look at what sort of objects are elements of $A \times (B \cup C)$. Since this is a Cartesian product on its outermost level, the objects in question will all be ordered pairs, each of whose abscissa[1] is an element of A with an ordinate[2] drawn from $BcupC$; that is to say, from B or C.

Now let's look at $(A \times B) \cup (A \times C)$; we hope that it will have elements meeting the same criteria. On its outermost level, it is not a Cartesian product but is a union; It is specifically a union, however, of Cartesian products. Thus the elements of this set are elements of either $A \times B$ or $A \times C$, both of which are sets of ordered pairs. Elements from either set will have abscissas drawn from the set A, while their ordinates might be from B or C. Thus this union describes the same set of ordered pairs as above.

[1] A high-falutin' term for the first coordinate in an ordered pair

[2] Same, but for the second coordinate