1. *Prove that for any natural number n, if n is odd and $n > 1$, then $\log_2 n$ is irrational. (hint: how can this logarithm be expressed arithmetically in terms exclusively of integers?)*

   Proof. Suppose counterfactually that $\log_2 n \in \mathbb{Q}$. Then there is an integer p and natural number q such that $\frac{p}{q} = \log_2 n$. Then, from the way a logarithm is defined, it must be the case that $n = 2^{p/q}$; raising both sides of this equation to the qth power gives $2^p = n^q$. Since $q \geq 1$, $n^q \geq n > 1$, so $2^p > 1$ and thus $p > 0$. Since p is an integer greater than zero, we can be certain that 2^p is even: $2^p = 2(2^{p-1})$. However, n^q is a product of q copies of the odd number n, and we know that products of odd numbers are odd from a previous result in the class, so n^q is odd. Since 2^p and n^q have different parities, they cannot possibly be equal, contradicting the assertion that $2^p = n^q$. □

2. *Prove that for any integers a and b, it is the case that $a^2 - 4b - 2 \neq 0$.*

   Proof. Suppose by way of contradiction that there are integers a and b such that $a^2 - 4b - 2 = 0$. Then $a^2 = 4b + 2 = 2(b + 1)$, so that a^2 is even. From a lemma presented in class (or a special case of a proposition on a previous problem set), if a^2 is even, then so is a; thus $a = 2k$ for some $k \in \mathbb{Z}$. Now the original counterfactual equation is $(2k)^2 - 4b - 2 = 0$, which can be algebraically rearranged to give $4(k^2 - b) = 2$. Since $k^2 - b$ is an integer, this ensures that $4 \mid 2$. But it is a known arithmetic fact that $4 \nmid 2$, leading to a contradiction. □

3. *Prove that there is no pair of real numbers whose sum is 30 and whose product is 500.*

   Proof. Suppose by way of contradiction that there exists a pair of real numbers a and b such that $a + b = 30$ and $ab = 500$. Note that squaring both sides of this first equation gives $a^2 + 2ab + b^2 = 900$; since $ab = 500$, we can simplify this to $a^2 + b^2 = -100$. However, it was proven in class that for any real number x, it is the case that $x^2 \geq 0$; thus $a^2 + b^2 \geq 0$, contradicting the requirement that it equal -100. □

   Alternative approach. Suppose by way of contradiction that there exists a pair of real numbers a and b such that $a+b = 30$ and $ab = 500$. Thus $a = 30-b$, so that $500 = ab = (30-b)b = 30b-b^2$. Rearranging this equation, we see than $b^2 - 30b + 225 = -275$; algebraically condensing the left side gives $(b - 15)^2 = -275$. However, since $b - 15$ is a real number, its square is nonnegative, leading to the arithmetic contradiction $-275 \geq 0$. □