1. *For the following investigations, consider relations R and R' on a set S such that $R \subseteq R' \subseteq S \times S$.*

   (a) *If R is reflexive, is it necessarily the case that R' is reflexive? Prove your assertion if true; furnish a counterexample if not.*

   This will always be true.

   Proof. Given that R is a reflexive relation on S, we shall seek to prove that R' is a reflexive relation on S; in other words, we wish to show that for any $x \in S$, it is true that $x\ R'\ x$.

   By reflexivity of R, we know that for any $x \in S$, it is the case that $x\ R\ x$, or in other words that $(x, x) \in R$. Since $R \subseteq R'$, we thus know that $(x, x) \in R'$, so that $x\ R'\ x$. □

   (b) *If R is symmetric, is it necessarily the case that R' is symmetric? Prove your assertion if true; furnish a counterexample if not.*

   This is not necessarily true. As a very simple case, consider $S = \{1, 2\}$, $R = \emptyset$, and $R' = \{(1, 2)\}$. R is a vacuously symmetric relation, and $R \subseteq R'$, but R' is nonsymmetric since $(1, 2) \in R'$ but $(2, 1) \notin R'$.

   Other, less vacuous examples are also possible.

   (c) *If R is transitive, is it necessarily the case that R' is transitive? Prove your assertion if true; furnish a counterexample if not.*

   This is not necessarily true. As a very simple case, consider $S = \{1, 2, 3\}$, $R = \emptyset$, and $R' = \{(1, 2), (2, 3)\}$. R is a vacuously symmetric relation, and $R \subseteq R'$, but R' is nonsymmetric since $(1, 2) \in R'$ and $(2, 3) \in R'$ but $(1, 3) \notin R'$.

   Other, less vacuous examples are also possible.

2. *Let $S = \{1, 2, 3, 4\}$. Either by explicitly constructing the relation as a set of ordered pairs, or by specifying the circumstances under which two elements of S are related, fully describe a relation satisfying each of the eight conditions below, and briefly explain why your relation satisfies the conditions.*

   (a) *A relation R_1 on S which is non-reflexive, non-symmetric, and non-transitive.*

   A minimal explicit construction would be $R_1 = \{(1, 2), (2, 3)\}$. This is nonreflexive since $(1, 1) \notin R_1$, nonsymmetric since $(1, 2) \in R_1$ but $(2, 1) \notin R_1$, and nontransitive since both $(1, 2)$ and $(2, 3)$ are in R_1 but $(1, 3)$ is not.

   A descriptive construction might be that, for instance, x is related to y when $y = x + 1$; this has the same arguments for satisfying the properties as are given above.

   (b) *A relation R_2 on S which is non-reflexive, non-symmetric, and transitive.*

   A minimal explicit construction would be $R_2 = \{(1, 2)\}$. This is nonreflexive since $(1, 1) \notin R_2$, and nonsymmetric since $(1, 2) \in R_2$ but $(2, 1) \notin R_2$. It will be vacuously transitive, however: since there are no a, b, and c suct that both (a, b) and (b, c) are in R_2, the premise of the transitivity condition is never met and the condition as a whole is vacuously true.

   A good descriptive construction, which works both for a relation on S and a relation on sets of at least two numbers generally, is to establish that x is related to y when $x < y$ (or, alternatively, when $x > y$). This is well-known to be a transitive relation, but is also explicitly non-symmetric and non-reflexive.

(c) *A relation R_3 on S which is non-reflexive, symmetric, and non-transitive.*

A minimal explicit construction would be $R_3 = \{(1,2),(2,1),(2,3),(3,2)\}$. This is nonreflexive since $(1,1) \notin R_3$ and nontransitive since both $(1,2)$ and $(2,3)$ are in R_3 but $(1,3)$ is not. However, it is symmetric since for each $(a,b) \in R$, it is also the case that (b,a) is an element of R_3.

A possible descriptive construction might be that x relates to y if and only if $|x-y| = 1$; the required difference means no x is related to itself, while the absolute value on the difference measurement guarantees symmetry, and the case of 1, 2, and 3 serves to illustrate non-transitivity.

A common relation matching these criteria is the symmetric inequality, that x relates to y when $x \neq y$.

(d) *A relation R_4 on S which is non-reflexive, symmetric, and transitive.*

Examples of this end up being necessarily very limited. In fact, if all four elements of S *appear* in a symmetric, transitive relation, then the relation is intrinsically reflexive. So only a relation in which at least one element of S does not participate at all can fit into this category. The simplest such would be $R_4 = \emptyset$, which is vacuously symmetric and transitive. If you desire a less vacuous example, something like $R_4 = \{(1,1),(1,2),(2,1),(2,2),(3,3)\}$ will do, since it is straightforwardly shown to be symmetric, exhaustively shown to be transitive, and is nonreflexive since $(4,4) \notin R_4$.

(e) *A relation R_5 on S which is reflexive, non-symmetric, and non-transitive.*

A minimal explicit construction would be $R_5 = \{(1,1),(1,2),(2,2),(2,3),(3,3),(4,4)\}$. This is reflexive since every $(x,x) \in R_5$, nonsymmetric since $(1,2) \in R_5$ but $(2,1) \notin R_5$, and nontransitive since both $(1,2)$ and $(2,3)$ are in R_5 but $(1,3)$ is not.

A descriptive construction might be that, for instance, x is related to y when $x \leq y \leq x+1$; this has the same arguments for satisfying the properties as are given above.

(f) *A relation R_6 on S which is reflexive, non-symmetric, and transitive.*

A minimal explicit construction would be $R_6 = \{(1,1),(1,2),(2,2),(3,3),(4,4)\}$. This is This is reflexive since every $(x,x) \in R_6$ and nonsymmetric since $(1,2) \in R_6$ but $(2,1) \notin R_6$. It will be transitive for near-vacuous reasons, however: since there are no distinct a, b, and c suct that both (a,b) and (b,c) are in R, the premise of the transitivity condition is only met by expressions of the form (a,a) and (a,c), or (a,c) and (c,c), and in both cases the conclusion is identical to a premise.

A good descriptive construction, which works both for a relation on S and a relation on sets of at least two numbers generally, is to establish that x is related to y when $x \leq y$ (or, alternatively, when $x \leq y$). This is well-known to be a transitive relation, and the nonstrictness of the inequality makes it reflexive, but it is also explicitly non-symmetric.

(g) *A relation R_7 on S which is reflexive, symmetric, and non-transitive.*

A minimal explicit construction would be $R_7 = \{(1,1),(1,2),(2,1),(2,2),(2,3),(3,2),(3,3),(4,4)\}$. This is reflexive since each $(x,x) \in R_7$ and nontransitive since both $(1,2)$ and $(2,3)$ are in R_7 but $(1,3)$ is not. However, it is symmetric since for each $(a,b) \in R_7$, it is also the case that (b,a) is an element of R_7.

A possible descriptive construction might be that x relates to y if and only if $|x-y| \leq 1$; since $x - x = 0$ every x is related to itself, while the absolute value on the difference measurement guarantees symmetry, and the case of 1, 2, and 3 serves to illustrate non-transitivity.

(h) *A relation R_8 on S which is reflexive, symmetric, and transitive.*

Any equivalence relation works here; $R_8 = \{(1,1),(2,2),(3,3),(4,4)\}$ is the smallest such.

A point of potential interest: suppose we were to consider *every* possible relation on this 4-element set. There are a lot of such relations, since there are $4^2 = 16$ relationships each of which could be present or absent, making $2^{16} = 65536$ possible relations! Here's a list of how they divide into the eight possible classes, with references. If you like this breakdown, you'll love MATH 387.

Reflexive?	Symmetric?	Transitive?	Count	Notes
Yes	Yes	Yes	15	Equivalencies of $\{1,2,3,4\}$; see OEIS A000110.
Yes	Yes	No	49	2^6 reflexive, symmetric relations in total; subtract out equivalences.
Yes	No	Yes	340	Non-equivalent quasi-orderings of $\{1,2,3,4\}$; see OEIS A000798.
Yes	No	No	3692	2^{12} reflexive relations in total; subtract out symmetric/transitive.
No	Yes	Yes	37	Equivalencies of proper subsets of $\{1,2,3,4\}$; see OEIS A000110.
No	Yes	No	923	2^{10} symmetric relations; subtract out reflexive/transitive.
No	No	Yes	3602	See OEIS A000110 for transitive relations; subtract symmetric/reflexive
No	No	No	56878	2^{16} relations in total; subtract out reflexive/symmetric/transitive.

So actually if we were to build a relation at random, say, by flipping a coin 16 times to determine if each of the 16 relationships hold, there's a very good likelihood (about 87%) that the relation we get would not be reflexive, symmetric, or transitive.

3. *Let R be a relation on $\mathbb{N}$ such that m relates to n if m and n have the same number of factors. In other, more technical words, we may explicitly define R as:*

$$R = \big\{(m,n) \in \mathbb{N} \times \mathbb{N} : |\{d \in \mathbb{N} : d \mid m\}| = |\{d \in \mathbb{N} : d \mid n\}|\big\}$$

(a) *Prove that R is an equivalence relation.*

Proof. $\square$

(b) *Describe two of the equivalence classes of R.*

An equivalence class consists of every natural number with some specific number of factors. In two cases these classes are very easily described: the set of all natural numbers with exactly one factor is the finite set $\{1\}$. The set of all natural numbers with exactly two factors are defenitionally the primes: $\{2,3,5,7,11,13,\ldots\}$.

A few more exotic ones are the set of natural numbers with exactly three factors (the squares of primes): $\{4,9,25,49,121,\ldots\}$, or, more convolutedly, those with exactly four factors (which are either the products of two primes *or* the cube of a prime): $\{6,8,10,14,15,21,22,26$