1. **(10 points)** *Consider the true statement "There is a natural number which is both even and prime." Write its negation as a quantified statement in words or symbols and briefly explain why this new statement must be false.*

   The negation of an existential is a universal, so in this case: "Every real number is either odd or nonprime", or, in symbols, $\forall x \in \mathbb{R}, (x \text{ is odd}) \vee (x \text{ is nonprime})$.

   This is not a true statement since 2 is neither odd nor nonprime.

2. **(18 points)** *Fill in the truth table for each of the following statements, and identify the statement as a tautology, a contradiction, or neither.*

   (a) $(P \vee Q) \leftrightarrow P$.

P	Q	$P \vee Q$	$(P \vee Q) \leftrightarrow P$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	T

   This statement is neither a contradiction nor a tautology.

   (b) $[(P \to Q) \wedge {\sim}Q] \to {\sim}P$.

P	Q	$P \to Q$	${\sim}Q$	$(P \to Q) \wedge {\sim}Q$	${\sim}P$	$[(P \to Q) \wedge {\sim}Q] \to {\sim}P$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

   Since this statement is always true, it is a tautology.

   (c) $Q \wedge {\sim}(P \to Q)$.

P	Q	$P \to Q$	${\sim}(P \to Q)$	$Q \wedge {\sim}(P \to Q)$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	F	F

   This statement is a contradiction, since it is always false.

3. **(22 points)** *Prove that for any real number x, if $x^3 + x < 10$, then $x < 2$.*

   Because this statement has a cumbersome premise and a simple conclusion, recasting it in terms of its contrapositive shows promise.

   Proof. Let us consider instead the contrapositive statement that if $x \geq 2$, then $x^3 + x \geq 10$. From our new premise, since $x \geq 2$, $x^3 \geq 8$, and thus $x^3 + x \geq 8 + 2 = 10$. □

4. **(11 points)** *The statement "for any real number x, if x is not an integer, then $7x$ is also not an integer" is false. State its converse. Is the converse true or not? Briefly explain your reasoning.*

   The converse of this statement is "for any real number x, if $7x$ is not an integer, then x is not an integer." This is a true statement; one easy way to show it is to look at its contrapositive: if x is an integer, then $7x$ is surely an integer.

5. **(22 points)** *Prove that for any integers m and n, if n is odd, then $nm - m^2$ is even.*

Since parity is clearly relevant to our conclusion but we are not given any guidance in the premise as to whether m is even or odd, we shall divide up into cases based on the parity of m.

Proof. We may note from our premise that since n is odd, there is an integer k such that $n = 2k + 1$. Then we consider two separate cases, depending on the parity of m.

Case I: m is even. Then by our case premise $m = 2\ell$ for some integer ℓ. Then,

$$nm - m^2 = (2k + 1)2\ell - (2\ell)^2 = 2(2k\ell + \ell - 2\ell^2)$$

demonstrating that the above expression is even.

Case I: m is odd. In this case, our case premise informs us that $m = 2\ell + 1$ for some integer ℓ. Then,

$$nm - m^2 = (2k+1)(2\ell+1) - (2\ell+1)^2 = 4k\ell + 2k + 2\ell + 1 - (4\ell^2 + 4\ell + 1) = 2(2k\ell + k - \ell - 2\ell^2)$$

and since $k\ell + k - \ell - 2\ell^2$ is an integer, the above expression is once again even.

□

6. **(22 points)** *Find sets satisfying the conditions below, or briefly explain why it is impossible to find such sets:*

(a) *Finite sets A and B such that $|A| = 4$, $|B| = 2$, and $B \in A$.*

We need a 4-element set A, one of whose elements is the two-element set B. A simple such example is $B = \{1, 2\}$ and $A = \{\{1, 2\}, 3, 4, 5\}$.

(b) *Sets R, S, and T such that $R \subsetneq S$, $S \subseteq T$, and $S \cap T = \emptyset$.*

This is actually impossible. If $R \subsetneq S$, it is impossible for S to be empty, so there is some $x \in S$, and since $S \subseteq T$, x is also an element of T, and then $S \cap T$ has at least one element (the aforementioned x).

(c) *Sets X, Y, and Z such that $X \in Y$, $X \subseteq Z$, and $Y \in Z$.*

We might start by letting $X = \{1, 2\}$; then the following rules would require $\{1, 2\} \in Y$, $1 \in Z$, and $2 \in Z$. The first of these conditions might lead us to choose $Y = \{\{1, 2\}, 3\}$; now we need Z to have 1, 2, and Y as elements, so we might select $Z = \Big\{1, 2, \{\{1, 2\}, 3\}\Big\}$.

A simpler solution — simpler to the point of triviality — would be $X = \emptyset$, $Y = \{\emptyset\}$, and $Z = \{\{\emptyset\}\}$.

7. **(20 points)** *Prove that for integers k and ℓ, if k is even and ℓ is odd, then $k\ell + 3\ell$ is odd.*

Since our premise is a conjunction of simple, useful statements, we shouldn't need to do anything too fancy here but can just write a direct proof.

Proof. From our premise that k is even and ℓ is odd, we may make use of the parity definitions to determine that $k = 2a$ and $\ell = 2b + 1$ for some integers a and b. Then

$$k\ell + 3\ell = 2a(2b + 1) + 3(2b + 1) = 4ab + 2a + 6b + 3 = 2(2ab + a + 3b + 1) + 1$$

and since $2ab + a + 3b + 1$ is an integer, the above expression is odd by definition. □