1. **(7 points)** *Consider the true statement "For every real number x, if $x > 2$, then $x^2 > 4$." Write its negation as a quantified statement in words or symbols and briefly explain why this new statement must be false.*

   The negation of a universal is an existential, so in this case: "There is a real number x such that $x > 2$ does not imply that $x^2 > 4$", or, in symbols, $\exists x \in \mathbb{R} : \sim(x > 2 \to x^2 > 4)$. We could simplify this further by noting that the negation of an implication is actually a conjunction (i.e. $\sim(P \to Q) \equiv (P \wedge \sim Q)$, so the existential above could be further rewritten as "There is a real number x such that $x > 2$ but x^2 is not greater than 4", or, in symbols, as $\exists x \in \mathbb{R} : x > 2 \wedge x^2 \leq 4$.

   This is not a true statement since, for any $x > 2$, we know that $x^2 > 2 \cdot 2$; thus it would surely be impossible to find a number larger than 2 whose square is less than 4, so the existential statement is false.

2. **(18 points)** *Find sets satisfying the conditions below, or briefly explain why it is impossible to find such sets:*

   (a) *Sets R, S, and T such that $R \in S$, $S \in T$, and $S \subsetneq T$.*

   We might conjecturally choose R to be $\{1\}$, as a nice simple example. Then, since $R \in S$, we would need S to be a set with $\{1\}$ as an element, and we might provisionally let $S = \big\{\{1\}\big\}$ (with the possibility that we would need to change it later. Finally, S needs to be both an element *and* a subset of T, which means T needs to have both S itself and S's element as elements. So T needs at least two elements: S itself, which will be $\big\{\{1\}\big\}$, and the sole element of S, which is $\{1\}$. So $T = \Big\{\big\{\{1\}\big\}, \{1\}\Big\}$.

   Other answers are also possible, of course. The simplest possibility, in the sense of making use of the smallest sets possible, is $R = \emptyset$, $S = \{\emptyset\}$, and $T = \big\{\emptyset, \{\emptyset\}\big\}$.

   (b) *Sets A and B such that $A \subseteq \mathbb{N}$, $B \subseteq \mathbb{N}$, and $A \in B$.*

   This is impossible. If B is a subset of $\mathbb{N}$, then all of B's elements are natural numbers, but the requirement that $A \in B$ would necessitate that at least one element of B would be a set.

   (c) *Finite sets X, Y, and Z such that $X \subseteq Z$, $Y \subseteq Z$, $|X| = |Y|$, and $|X \cap Y| = 2$.*

   The requirement here is to simply produce two sets X and Y, of the same finite size, and with an overlap of 2 elements, and then we can just choose any Z large enough to contain both of them. An easy way to do this is to let $X = Y = Z = \{1, 2\}$. If you want to be less absurd, something like $X = \{1, 2, 3, 5, 7\}$, $Y = \{1, 2, 4, 6, 8\}$, and $Z = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ would do.

3. **(19 points)** *Prove that for any integer n, if $n^2 + 4n + 1$ is even, then n is odd.*

   Because this statement has a cumbersome premise and a simple conclusion, recasting it in terms of its contrapositive shows promise.

   Proof. Let us consider instead the contrapositive statement that if n is even, then $n^2 + 4n + 1$ is odd. From our new premise that n is even, we may expand the definition of even to assert that $n = 2k$ for some integer k. Then

   $$n^2 + 4n + 1 = (2k)^2 + 4(2k) + 1 = 4k^2 + 8k + 1 = 2(2k^2 + 4k) + 1$$

   and since $2k^2 + 4k$ is an integer, the above expression is definitionally odd. □

4. **(8 points)** *The statement "for any natural number n, if n has an odd number of factors, then it is a perfect square" is true. State its converse. Is the converse true or not? Briefly explain your reasoning.*

The converse of this statement is "for any natural number n, if n is a perfect square, then it has an odd number of factors." This is a true statement. One easy way to see it to be true is that *most* of the factors of n^2 come in pairs: whenever $ab = n^2$, both a and b are factors. The one exception to this factorization contributing two factors to the complete list of factors is that $nn = n^2$, which contributes only a single factor to the list. So when we write out, for instance, all the ways to write 36 as a product of two numbers we have $1 \cdot 36$, $2 \cdot 18$, $3 \cdot 12$, $4 \cdot 9$, and $6 \cdot 6$. The first four products describe two factors each, while the last describes only one—and because of that last, idiosyncratic factorization, the total count of factors is odd.

5. **(15 points)** *Prove that for integers a, b, and c, if a and c are both odd, then $ab + bc$ is even.*

Since our premise is a conjunction of simple, useful statements, we shouldn't need to do anything too fancy here but can just write a direct proof.

Proof. From our premise that a and c are both odd, we may make use of the definition of odd to determine that $a = 2k + 1$ and $c = 2\ell + 1$ for some integers k and ℓ. Then

$$ab + bc = (2k + 1)b + b(2\ell + 1) = b(2k + 2\ell + 2) = 2(kb + \ell b + b)$$

and since $kb + \ell b + b$ is an integer, the above expression is even by definition. □

6. **(15 points)** *Fill in the truth table for each of the following statements, and identify the statement as a tautology, a contradiction, or neither.*

(a) $P \wedge (Q \to \sim P)$.

P	Q	$\sim P$	$Q \to \sim P$	$P \wedge (Q \to \sim P)$
F	F	T	T	F
F	T	T	T	F
T	F	F	T	T
T	T	F	F	F

This statement is neither a contradiction nor a tautology.

(b) $(P \wedge Q) \leftrightarrow (P \to \sim Q)$.

P	Q	$P \wedge Q$	$\sim Q$	$P \to \sim Q$	$(P \wedge Q) \leftrightarrow (P \to \sim Q)$
F	F	F	T	T	F
F	T	F	F	T	F
T	F	F	T	T	F
T	T	T	F	F	F

Since this statement is always false, it is a contradiction.

(c) $(P \to Q) \to (Q \to P)$.

P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \to (Q \to P)$
F	F	T	T	T
F	T	T	F	F
T	F	F	T	T
T	T	T	T	T

This statement is neither a contradiction nor a tautology.

7. **(18 points)** *Prove that for any integer n, $3n^2 - 5n$ is even.*

Since we have no premise provided, and since parity is clearly relevant to our conclusion, we shall divide up into cases based on the parity of n.

Proof. We shall consider two separate cases, depending on the parity of n.

Case I: n is even. Then by our case premise $n = 2k$ for some integer k. Then,

$$3n^2 - 5n = 3(2k)^2 - 5(2k) = 12k^2 - 10k = 2(6k^2 - 5k)$$

and since $6k^2 - 5k$ is an integer, the above expression is even.

Case I: n is odd. In this case, our case premise informs us that $n = 2k + 1$ for some integer k. Then,

$$3n^2 - 5n = 3(2k + 1)^2 - 5(2k + 1) = 12k^2 + 2k - 2 = 2(6k^2 + k - 1)$$

and since $6k^2 + k - 1$ is an integer, the above expression is once again even.

□