1. (a) **(16 points)** *Prove that there is no pair of sets A and B such that $A \cup B \subseteq A - B$ and $B \neq \emptyset$.*

Proof. Let us assume, counterfactually, that there is a set A and a nonempty set B such that $A \cup B \subseteq A - B$. Since B is nonempty, we can consider a specific $x \in B$. Since x is an element of B, it is an element of any union containing B; thus $x \in A \cup B$. Since $A \cup B \subseteq A - B$, it follows that $x \in A - B$, so $x \in A$ and $x \notin B$, contradicting our establishment of x as an element of B. □

Alternative contradiction-hater's proof. We may rephrase our proposition by turning our negated existential into a universal of a negation, and rewriting a negated conjunction as an implication so as to get the statement: "for any sets A and B, if B is nonempty, then $A \cup B \nsubseteq A - B$."

Since B is nonempty, there is an elment x of B. Since $x \in B$, it follows from the definition of a union that $x \in A \cup B$, and from the definition of a difference $x \notin A - B$. Since $A \cup B$ contains an element (namely, x) which is not in $A - B$, it follows that $A \cup B$ is not a subset of $A - B$. □

(b) **(4 points)** *When $B = \emptyset$, it is in fact the case that $A \cup B \subseteq A - B$. What aspect of your proof would fail to be valid when $B = \emptyset$?*

Our proof (either of the above proofs) was based on exploring the membership qualities of an element x of B; if B were empty, there would be no such element to explore.

2. **(10 points)** Disprove *the following statement: for any natural numbers a and b, the sum $a + b$ is less than or equal to the product ab.*

Disproof. Consider the counterexample $a = 1$ and $b = 2$; $1 + 2$ is greater than $1 \cdot 2$. □

3. **(15 points)** *Prove that for any integers n, a, b, and x, if $n \mid a$ and $n \mid b$, then $n \mid ax + b$.*

Proof. From our premises, $a = kn$ and $b = \ell n$ for some $k, \ell \in \mathbb{Z}$. Then

$$ax + b = knx + \ell n = (kx + \ell)n$$

and since $kx + \ell$ is an integer, $n \mid ax + b$. □

4. **(20 points)** *Prove that for any sets A, B, and C, if $A \subseteq B$, then $A - C \subseteq B - C$.*

Proof. Consider $x \in A - C$, which we shall seek to show is also an element of $B - C$. By the definition of a set difference, $x \in A$ and $x \notin C$. Then, because $A \subseteq B$ and $x \in A$, we may deduce that $x \in B$. Thus, since $x \in B$ and $x \notin C$, we see that $x \in B - C$. □

Alternative contradiction-lovers' proof. Let us assume by way of contradiction that $A \subseteq B$ but $A - C \nsubseteq B - C$. Since $A - C$ is not a subset of $B - C$, there must be some element x of $A - C$ which is not an element of $B - C$; since $x \in A - C$, it follows that $x \in A$ and $x \notin C$; since $x \notin B - C$, it must be the case that either $x \notin B$ or $x \in C$. We could consider these two separate possibilities, both of which lead to contradiction:

Case I: $x \notin B$**.** It was established from $x \in A - C$ that $x \in A$; thus, since $x \notin B$, we have established that x is specifically an element of A which is not an element of B, so $A \not\subseteq B$, contradicting our premise that $A \subseteq B$.

Case II: $x \notin C$**.** It was established from $x \in A - C$ that $x \notin C$, which directly contradicts this case premise. □

5. **(20 points)** *Prove that for any natural number n, it is the case that $3 \mid n^3 - n$.*

Proof. Let us divide our proposition into three cases, depending on n's remainder on division by 3:

Case I: $n = 3k$ **for some integer** k**.** Then $n^3 - n = 27k^3 - 3k = 3(9k^3 - k)$, so $3 \mid n^3 - n$.

Case II: $n = 3k + 1$ **for some integer** k**.** Then $n^3 - n = (27k^3 + 27k^2 + 9k + 1) - (3k + 1) = 3(9k^3 + 9k^2 + 2k)$, so $3 \mid n^3 - n$.

Case III: $n = 3k + 2$ **for some integer** k**.** Then $n^3 - n = (27k^3 + 54k^2 + 36k + 8) - (3k + 2) = 3(9k^3 + 18k^2 + 11k + 2)$, so $3 \mid n^3 - n$. □

Alternative proof for those who prefer factorization to expansion. We may note that $n^3 - n = (n - 1)n(n + 1)$ and divide our proposition into three cases, depending on n's remainder on division by 3:

Case I: $3 \mid n$**.** Then $n = 3k$ for some integer k, and multiplying both sides by $(n - 1)(n + 1)$ gives $n^3 - n = 3[k(n - 1)(n + 1)]$; thus $3 \mid n^3 - n$.

Case II: $3 \mid n + 1$**.** Then $n + 1 = 3k$ for some integer k, and multiplying both sides by $n(n - 1)$ gives $n^3 - n = 3[kn(n - 1)]$; thus $3 \mid n^3 - n$.

Case III: $3 \mid n + 2$**.** Then since $3 \mid 3$ we could use the result of Question 3 to note that $3 \mid (-1)3 + (n + 2)$ so $3 \mid n - 1$, which may be interpreted as an assertion that $n - 1 = 3k$ for some integer k. Then, multiplying both sides by $n(n + 1)$ gives $n^3 - n = 3[kn(n + 1)]$; thus $3 \mid n^3 - n$. □

6. **(20 points)** *Prove that for any positive integer n, it is the case that $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1) = \frac{n(n+1)(n+2)}{3}$.*

Proof. We proceed by induction; note that the base case $n = 1$ is trivially true, as $1 \cdot 2 = 2 = \frac{1 \cdot 2 \cdot 3}{3}$. Now, we assume, for a specific n, that it is true that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1) = \frac{n(n + 1)(n + 2)}{3}$$

and seek to prove that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1) + (n + 1)(n + 2) = \frac{(n + 1)(n + 2)(n + 3)}{3}$$

To do so, we may add $(n+1)(n+2)$ to both sides of our inductive hypothesis, and proceed via arithmetic:

$$\begin{aligned} 1\cdot 2+2\cdot 3+3\cdot 4+\cdots+n(n+1)+(n+1)(n+2) &= \frac{n(n+1)(n+2)}{3}+(n+1)(n+2) \\ &= \frac{n(n+1)(n+2)+3(n+1)(n+2)}{3} \\ &= \frac{(n+3)(n+1)(n+2)}{3} = \frac{(n+1)(n+2)(n+3)}{3} \end{aligned}$$

$\square$