1. **(10 points)** *Answer the following questions.*

   (a) **(5 points)** *For the function $g(x) = \frac{2}{\sqrt{x}} - 6x^4 + 2x$, calculate its second derivative $g''(x)$.*
   Taking two derivatives successively:

$$g'(x) = \frac{d}{dx}\left(2x^{-1/2} - 6x^4 + 2x\right) = -x^{-3/2} - 24x^3 + 2$$

$$f''(x) = \frac{d}{dx}\left(-x^{-3/2} - 24x^3 + 2\right) = \frac{3}{2x^{5/2}} - 72x^2$$

   (b) **(5 points)** *The height of a skydiver in meters t seconds after jumping out of a plane is given by the formula $14000 - 5t^2 + 2t\sqrt{t}$. How quickly is this skydiver moving after 9 seconds?*
   The velocity as a function is $\frac{d}{dt}(14000 - 5t^2 + 2t^{3/2}) = -10t + 3\sqrt{t}$, so after nine seconds their velocity is $-10 \cdot 9 + 3\sqrt{9} = -81$ (i.e., 34 meters per second downwards).

2. **(9 points)** *Let* $g(x) = \begin{cases} \sqrt[3]{x} & \text{if } x \leq 8 \\ ax & \text{if } 8 < x \leq 12 \\ \sqrt{x+b} & \text{if } x > 12 \end{cases}$.
   What choices of a and b will make this function continuous everywhere?

   The individual pieces of this function are themselves continuous, so we need only ensure continuity at the boundaries between the pieces, namely, at $x = 8$ and $x = 12$. At $x = 8$ we want the two functions $\sqrt[3]{x}$ and ax to coincide; their specific values at this point are 2 and $8a$ respectively. In order for these to be equal, it must be the case that $2 = 8a$, or in other words that $a = \frac{1}{4}$. Similarly, at $x = 12$, we want the functions $\frac{1}{4}x$ and $\sqrt{x+b}$ to coincide, which requires that $3 = \sqrt{12+b}$, so $b = -3$.

3. **(13 points)** *Let $f(x) = 5x^2 - 3x - 7$.*

   (a) **(9 points)** Using the difference quotient*, determine the formula for $f'(x)$.*

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{[5(x+h)^2 - 3(x+h) - 7] - (5x^2 - 3x - 7)}{h} \\ &= \lim_{h\to 0} \frac{(5x^2 + 10hx + 5h^2 - 3x - 3h - 7) - (5x^2 - 3x - 7)}{h} \\ &= \lim_{h\to 0} \frac{10xh + 5h^2 - 3h}{h} = \lim_{h\to 0} 10x + 5h - 3 = 10x - 3 \end{aligned}$$

   (b) **(4 points)** *Find the equation of the tangent line to $f(x)$ at the point $(-2, 19)$.*
   The slope of the tangent line is $f'(-2) = 10(-2) - 3 = -23$, and the equation of a line with slope -23 passing through the point $(-2, 19)$ is

$$y - 19 = -23(x + 2)$$

   or, if slope-intercept form is preferred,

$$y = -23x - 27.$$

4. **(9 points)** *Let $f(x) = 8x - 3$.*

   (a) **(1 point)** *Find $\lim_{x\to -1} f(x)$.*

   Since this function is a polynomial, $\lim_{x\to -1} f(x) = f(-1) = -11$.

   (b) **(8 points)** *For a choice of $\epsilon = 0.2$, what value of δ suffices to satisfy the conditions set by the epsilon-delta conception of the limit? Show your work or justify your answer.*

   The choice $\epsilon = 0.2$ requires us to get $8x - 3$ to within a distance of 0.2 of -11; that is to say, to get it to be between -11.2 and -10.8. This we can simplify to some required boundaries on x:

$$\begin{aligned} -11.2 < 8x - 3 &< -10.8 \\ -8.2 < 8x \quad &< -7.8 \\ -1.025 < \ x \quad &< -0.975 \end{aligned}$$

   so x must be within a distance of 0.025 of -1; this can be done by establishing a value of 0.025 (or anything smaller) for δ (which is the constraint on x's distance from -1).

5. **(24 points)** *Evaluate the following limits; when a limit can not be evaluated, explicitly say so or discuss its behavior.*

   (a) **(4 points)** $\lim_{u\to+\infty} \frac{1-2u^2-u^4}{5+u+3u^4}$.

   Dividing both numerator and denominator by u^4 to make the denominator finite, we find that

$$\lim_{u\to+\infty} \frac{1-2u^2-u^4}{5+u+3u^4} = \lim_{u\to+\infty} \frac{\frac{1-2u^2-u^4}{u^4}}{\frac{5+u+3u^4}{u^4}} = \lim_{u\to+\infty} \frac{\frac{1}{u^4} - \frac{2}{u^2} - 1}{\frac{5}{u^4} + \frac{1}{u^3} + 3} = \frac{0-0-1}{0+0+3} = \frac{-1}{3}.$$

   Alternatively, we might note that the numerator and denominator are dominated by $-u^4$ and $3u^4$ respectively as u grows very large, so the quotient tends towards $\frac{-u^4}{3u^4} = \frac{-1}{3}$.

   (b) **(4 points)** $\lim_{x\to 1} e^{x^3-x}$.

   This can be directly evaluated: $e^{1^3-1} = e^0 = 1$.

   (c) **(4 points)** $\lim_{t\to -3} \frac{t^2-9}{t^2+2t-3}$.

   Since direct evaluation yields $\frac{0}{0}$, we will need to factor and cancel a term of $(t+3)$ from both the numerator and denominator:

$$\lim_{t\to -3} \frac{t^2-9}{t^2+2t-3} = \lim_{t\to -3} \frac{(t+3)(t-3)}{(t+3)(t-1)} = \lim_{t\to -3} \frac{t-3}{t-1} = \frac{-6}{-4} = \frac{3}{2}.$$

   (d) **(4 points)** $\lim_{s\to -\infty} \frac{3-s^2}{2s^3+4s-3}$.

   Dividing both numerator and denominator by s^3 to make the denominator finite, we find that

$$\lim_{s\to -\infty} \frac{3-s^2}{2s^3+4s-3} = \lim_{s\to -\infty} \frac{\frac{3-s^2}{s^3}}{\frac{2s^3+4s-3}{s^3}} = \lim_{s\to -\infty} \frac{\frac{3}{s^3} - \frac{1}{s}}{2 + \frac{4}{s^2} - \frac{3}{s^3}} = \frac{0-0}{2+0-0} = 0.$$

Alternatively, we might note that the numerator and denominator are dominated by $-s^2$ and $2s^3$ respectively as s becomes very large in magnitude, so the quotient tends towards $\frac{-s^2}{2s^3} = \frac{-1}{s}$, which approaches 0 as s grows very large in magnitude.

(e) **(4 points)** $\lim\limits_{x\to 4} \frac{x^2+3x}{x^2-x-12}$.

Direct evaluation yields $\frac{28}{0}$, so this fails to exist in some sort of infinite/"blow-up" manner.

(f) **(4 points)** $\lim\limits_{r\to\infty} \frac{r^4-3r^2+r}{r^3-r+2}$.

Dividing both numerator and denominator by r^3 to make the denominator finite, we find that

$$\lim_{r\to\infty} \frac{r^4-3r^2+r}{r^3-r+2} = \lim_{r\to\infty} \frac{\frac{r^4-3r^2+r}{r^3}}{\frac{r^3-r+2}{r^3}} = \lim_{r\to\infty} \frac{r - \frac{3}{r} + \frac{1}{r^2}}{1 - \frac{1}{r^2} + \frac{2}{r^3}} = \lim_{r\to\infty} \frac{r-0+0}{1-0+0}$$

which increases without bound as r increases, so the limit cannot be evaluated (specifically, it "equals $+\infty$" in an idiomatic sense).

6. **(16 points)** *Answer the following questions.*

(a) **(6 points)** *Given* $f(u) = \frac{e^u+u^4}{u-3u^2}$, *find* $f'(u)$.

This expression is a quotient, so its derivative can be calculated with the quotient rule:

$$\begin{aligned} f'(u) &= \frac{d}{du}\frac{e^u+u^4}{u-3u^2} \\ &= \frac{(u-3u^2)\frac{d}{du}(e^u+u^4) - (e^u+u^4)\frac{d}{du}(u-3u^2)}{(u-3u^2)^2} \\ &= \frac{(u-3u^2)(e^u+4u^3) - (e^u+u^4)(1-6u)}{(u-3u^2)^2} \end{aligned}$$

(b) **(6 points)** *Determine* $\frac{d}{ds}\left[4e^s\left(\sqrt{s}+\frac{1}{s}\right)\right]$.

This expression is a product, so its derivative can be calculated with the product rule:

$$\begin{aligned} \frac{d}{ds}\left[4e^s\left(\sqrt{s}+\frac{1}{s}\right)\right] &= \left(\frac{d}{ds}4e^s\right)\left(\sqrt{s}+\frac{1}{s}\right) + 4e^s\frac{d}{ds}\left(s^{1/2}+s^{-1}\right) \\ &= 4e^s\left(\sqrt{s}+\frac{1}{s}\right) + 4e^s\frac{d}{ds}\left(\frac{1}{2}s^{-1/2}-s^{-2}\right) \\ &= 4e^s\left(\sqrt{s}+\frac{1}{2\sqrt{s}}+\frac{1}{s}-\frac{1}{s^2}\right) \end{aligned}$$

The last step serves to simplify the answer and is not necessary.

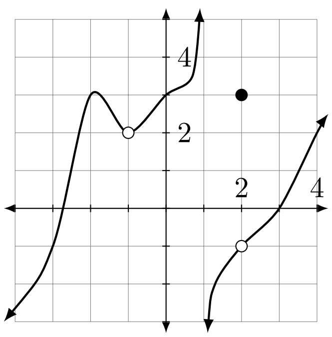
(c) **(4 points)** *For* $y = 4x^3 - 7\sqrt[3]{x} + 2 + \frac{6}{x^3}$, *find* $\frac{dy}{dx}$.

This can be differentiated term-by term, because each term is either a power function or an exponential function:

$$\frac{dy}{dx} = \frac{d}{dx}\left(4x^3 - 7x^{1/3} + 2 + 6x^{-3}\right) = 12x^2 - \frac{7}{3}x^{-2/3} - 18x^{-4} = 12x^2 - \frac{7}{3x^{2/3}} - \frac{18}{x^4}.$$

The last step is purely cosmetic and is not necessary.

7. **(9 points)** *For the plot of $f(x)$ shown below, indicate whether or not each of the following quantities can be evaluated. If they can be evaluated, compute their values. If they cannot be evaluated, give an idiomatic "value" if possible, and if not, explicitly say so. You need not show work.*



$\lim_{x\to -3} f(x)$

$f(-3)$

$\lim_{x\to -2} f(x)$

$f(-2)$

$\lim_{x\to 2^+} f(x)$

$\lim_{x\to 2} f(x)$

$\lim_{x\to 3} f(x)$

$f(3)$

$\lim_{x\to +\infty} f(x)$

You might notice this problem is not actually all that interesting; I goofed and changed the graph without updating the questions I asked about it! Most of the points you're asked to query are places where the function is actually continuous. This wasn't meant to be a "trick": it really is as straightforward as it ends up looking.

$\lim_{x\to -3} f(x)$ and $f(-3)$ are both -1, since the function is continuous at $(-3,-1)$.

$\lim_{x\to -2} f(x)$ and $f(-2)$ are both 3, since the function is continuous at $(-2,3)$.

$\lim_{x\to 2^+} f(x) = -1$, since the curve has y-values close to (and slightly below) -1 when x is slightly below 2.

$\lim_{x\to 2} f(x) = -1$, since the curve has y-values close to -1 when x is close to but not at 2.

$\lim_{x\to 3} f(x)$ and $f(3)$ are both 0, since the function is continuous at $(3,0)$.

$\lim_{x\to +\infty} f(x)$ is not well described here. It looks like $+\infty$, but this particular function is not drawn in a way which makes that obvious.

8. **(10 points)** *Given the function $f(x) = \frac{5-2x^3}{x^2-3x}$, answer the following questions.*

   (a) **(3 points)** *On which intervals is the function continuous?*

   The function is built mostly from pieces which are continuous on their whole domain; the one point of caution is that the denominator of a fraction cannot be zero, so the function is unevaluatable when $x^2 - 3x = 0$; this occurs when $x = 0$ or $x = 3$, and so the function is continuous at all points where $x \neq 0, 3$, or, alternatively, the intervals $(-\infty, 0), (0, 3), (3, \infty)$.

   (b) **(7 points)** *Describe, either in words or symbolically, the long-term behavior of the function in each direction.*

   For very large positive or negative values of x, $f(x)$ will be dominated in the numerator by the $-2x^3$ term and in the denominator by x^2. Thus for x of large magnitude, $f(x)$ resembles the function $\frac{-2x^3}{x} = -2x$.A s $x \to +\infty$, this expression decreases without bound (or "approaches $-\infty$"); as $x \to -\infty$, it expression increases without bound (or "approaches $+\infty$").