1. **(13 points)** *Let $f(x) = 8 - 2x - 3x^2$.*

   (a) **(9 points)** Using the difference quotient, *determine the formula for $f'(x)$.*

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{[8 - 2(x+h) - 3(x+h)^2] - (8 - 2x - 3)}{h} \\ &= \lim_{h\to 0} \frac{(8 - 2x - 2h - 3x^2 - 6xh - 3h^2) - (8 - 2x - 3)}{h} \\ &= \lim_{h\to 0} \frac{-2h - 6xh - 3h^2}{h} = \lim_{h\to 0} -2 - 6x - 3h = -2 - 6x \end{aligned}$$

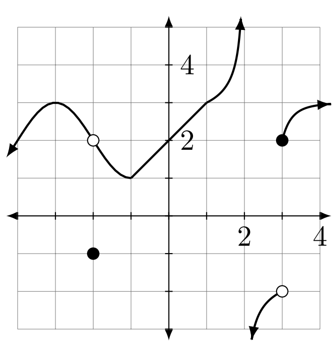
   (b) **(4 points)** *Find the equation of the tangent line to $f(x)$ at the point $(2, -8)$.*
   The slope of the tangent line is $f'(2) = -2 - 6 \cdot 2 = -14$, and the equation of a line with slope -14 passing through the point $(2, -8)$ is

$$y + 8 = -14(x - 2)$$

2. **(9 points)** *Let* $f(x) = \begin{cases} px & \text{if } x \leq -3 \\ x^2 - 2 & \text{if } -3 < x < 2 \\ x^3 + q & \text{if } x \geq 2 \end{cases}$.
   What choices of p and q will make this function continuous everywhere?

   The individual pieces of this function are themselves continuous, so we need only ensure continuity at the boundaries between the pieces, namely, at $x = -3$ and $x = 2$. At $x = -3$ we want the two functions px and $x^2 - 2$ to coincide; their specific values at this point are $-3p$ and 7 respectively. In order for these to be equal, it must be the case that $-3p = 7$, or in other words that $p = \frac{-7}{3}$. Similarly, at $x = 2$, we want the functions $x^2 - 2$ and $x^3 + 2$ to coincide, which requires that $2 = 8 + q$, so $q = -6$.

3. **(9 points)** *For the plot of $f(x)$ shown below, indicate whether or not each of the following quantities can be evaluated. If they can be evaluated, compute their values. If they cannot be evaluated, give an idiomatic "value" if possible, and if not, explicitly say so. You need not show work.*



$\lim_{x\to -3} f(x)$

$f(-3)$

$\lim_{x\to -2} f(x)$

$f(-2)$

$\lim_{x\to 2^+} f(x)$

$\lim_{x\to 2} f(x)$

$\lim_{x\to 3} f(x)$

$f(3)$

$\lim_{x\to +\infty} f(x)$

$\lim_{x\to -3} f(x)$ is determined to be 3 because as the curve approaches $x = -3$ from the either the left or right, its y-coordinate rises toward 3.

$f(-3) = 3$, because the curve passes through $(3, 3)$.

$\lim_{x\to -2} f(x) = 2$, because as the curve approaches $x = -2$ from either the left or right, its y-coordinateapproaches 2.

$f(-2) = -1$, because of the black dot at $(-2, -1)$

$\lim_{x\to 2^+} f(x) = +\infty$, because as the curve approaches $x = 2$ from the right, its y-coordinate decreases without bound.

$\lim_{x\to 2} f(x)$ does not exist, because the behaviors to the immediate left and right of $x = 2$ do not match.

$\lim_{x\to 3} f(x)$ does not exist, because $\lim_{x\to 3^-} f(x)$ and $\lim_{x\to 3^+} f(x)$ both exist but are not equal.

$f(3) = 2$, because of the black dot at $(3, 2)$.

$\lim_{x\to +\infty} f(x)$ is 3, because as we follow the curve off to the right, it approaches the y-coordinate of 3.

4. **(10 points)** *Given the function* $f(x) = \frac{8x^2 \arctan x - 7x}{x^2 - 7x + 12}$, *answer the following questions preparatory to sketching the functions.*

   (a) **(3 points)** *On which intervals is the function continuous?*

   The function is built mostly from pieces which are defined everywhere; the one point of caution is that the denominator of a fraction cannot be zero, so the function is unevaluatable when $x^2 - 7x + 12 = 0$; this occurs when $x = 3$ or $x = 4$, and so the domain is all points where $x \neq 3, 4$, or, alternatively, the intervals $(-\infty, 3), (3, 4), (4, \infty)$.

   (b) **(7 points)** *Describe, either in words or symbolically, the long-term behavior of the function in each direction.*

   For very large positive or negative values of x, $f(x)$ will be dominated in the numerator by the $8x^2 \arctan x$ term and in the denominator by x^2. Thus for x of large magnitude, $f(x)$ resembles the function $8 \arctan x$. Note that $\lim_{x\to+\infty} \arctan x = \frac{\pi}{2}$ and $\lim_{x\to-\infty} \arctan x = \frac{-\pi}{2}$, so over the long term $f(x)$ will approach 4π and -4π to the right and left respectively.

5. **(10 points)** *Answer the following questions.*

   (a) **(5 points)** *A ball is rolling down a hill such that its position along the hill after t seconds is $3t^2 + 5t - 2\sqrt{t} + 4$ meters. What is its* velocity *after 4 seconds?*

   The velocity as a function is $\frac{d}{dt}(3t^2 + 5t - 2t^{1/2} + 4) = 6t + 5 - t^{-1/2}$, so after four seconds its velocity is $6 \cdot 4 + 5 - \frac{1}{2} = \frac{57}{2}$.

   (b) **(5 points)** *For the function $f(x) = 2x^5 - 7e^x + \frac{2}{x}$, calculate its second derivative $f''(x)$.*

   Taking two derivatives successively:

$$f'(x) = \frac{d}{dx}\left(2x^5 - 7e^x + 2x^{-1}\right) = 10x^4 - 7e^x - 2x^{-2}$$

$$f''(x) = \frac{d}{dx}\left(10x^4 - 7e^x - 2x^{-2}\right) = 40x^3 - 7e^x + 4x^{-3}$$

6. **(16 points)** *Answer the following questions.*

(a) **(4 points)** *Given* $f(r) = 3r^5 - 2e^r + \frac{7}{\sqrt{r}}$, *find* $f'(r)$.

This can be differentiated term-by term, because each term is either a power function or an exponential function:

$$f'(r) = \frac{d}{dr}\left(3r^5 - 2e^r + 7r^{-1/2}\right) = 15r^4 - 2e^r - \frac{7}{2}r^{-3/2} = 15r^4 - 2e^r - \frac{7}{2r^{3/2}}.$$

The last step is purely cosmetic and is not necessary.

(b) **(6 points)** *Determine* $\frac{d}{dt}\frac{t^4-2t+1}{e^t-7t^3}$.

This expression is a quotient, so its derivative can be calculated with the quotient rule:

$$\begin{aligned}\frac{d}{dt}\frac{t^4-2t+1}{e^t-7t^3} &= \frac{(e^t-7t^3)\frac{d}{dt}(t^4-2t+1) - (t^4-2t+1)\frac{d}{dt}(e^t-7t^3)}{(e^t-7t^3)^2} \\ &= \frac{(e^t-7t^3)(4t^3-2) - (t^4-2t+1)(e^t-21t^2)}{(e^t-7t^3)^2}\end{aligned}$$

(c) **(4 points)** *For* $y = (e^x - 7x^2)(e^x + 12\sqrt{x})$, *find* $\frac{dy}{dx}$.

This expression is a product, so its derivative can be calculated with the product rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left[(e^x - 7x^2)(e^x + 12\sqrt{x})\right] \\ &= \left[\frac{d}{dx}(e^x - 7x^2)\right](e^x + 12\sqrt{x}) + (e^x - 7x^2)\frac{d}{dx}(e^x + 12x^{1/2}) \\ &= (e^x - 14x)(e^x + 12\sqrt{x}) + (e^x - 7x^2)(e^x + 6x^{-1/2})\end{aligned}$$

7. **(9 points)** *Let* $g(t) = 13 - \frac{3t}{2}$.

(a) **(1 point)** *Find* $\lim_{t\to 4} g(t)$.

Since this function is a polynomial, $\lim_{t\to 4} g(t) = g(4) = 7$.

(b) **(8 points)** *Using epsilon-delta methods, justify your result above.*

The statement $\lim_{t\to 4} 13 - \frac{3t}{2} = 7$ is an assertion that, for every value $\epsilon > 0$, a value δ can be furnished such that, if $0 < |x-4| < \delta$, then $|13 - \frac{3t}{2} - 7| < \epsilon$. We may justify this assertion by explicity determining how δ is calculated from ϵ to make this inference true.

$$\begin{aligned}\left|13 - \frac{3t}{2} - 7\right| &< \epsilon \\ \left|6 - \frac{3t}{2}\right| &< \epsilon \\ \left|\frac{-3}{2}(t-4)\right| &< \epsilon \\ \frac{3}{2}\left|(t-4)\right| &< \epsilon \\ |t-4| &< \frac{2\epsilon}{3}\end{aligned}$$

so we may declare that the choice of δ equal to $\frac{2\epsilon}{3}$ is sufficient to meet whatever challenge we are given.

8. **(24 points)** *Evaluate the following limits; when a limit can not be evaluated, explicitly say so or discuss its behavior.*

   (a) **(4 points)** $\lim_{t\to 9} \frac{t^3-\sqrt{t}}{t^2-10t+9}$.

   Since direct evaluation yields $\frac{726}{0}$, this limit does not exist (in some infinite/“blow-up” manner).

   (b) **(4 points)** $\lim_{y\to -3} \frac{2y^2-18}{y^2-y-12}$.

   Since direct evaluation yields $\frac{0}{0}$, we will need to factor and cancel a term of $(x-3)$ from both the numerator and denominator:

$$\lim_{y\to-3}\frac{2y^2-18}{y^2-y-12} = \lim_{y\to-3}\frac{(2y-6)(y+3)}{(y-4)(y+3)} = \lim_{y\to-3}\frac{2y-6}{y-4} = \frac{-12}{-7} = \frac{12}{7}.$$

   (c) **(4 points)** $\lim_{z\to+\infty} \frac{2+4z^2-8z^4}{2z^5-4z^3}$.

   Dividing both numerator and denominator by z^5 to make the denominator finite, we find that

$$\lim_{z\to+\infty}\frac{2+4z^2-8z^4}{2z^5-4z^3} = \lim_{z\to+\infty}\frac{frac2z^5+\frac{4}{z^3}-\frac{8}{z}}{2-\frac{4}{z^2}} = \frac{0+0-0}{2-0} = 0.$$

   (d) **(4 points)** $\lim_{s\to-\infty} \frac{s^9-4s+1}{13-4s^9}$.

   Dividing both numerator and denominator by s^9 to make the denominator finite, we find that

$$\lim_{s\to-\infty}\frac{s^9-4s+1}{13-4s^9} = \lim_{s\to-\infty}\frac{1-\frac{4}{s^8}+\frac{1}{s^9}}{\frac{13}{s^9}-4} = \frac{1-0+0}{0-4} = \frac{-1}{4}.$$

   (e) **(4 points)** $\lim_{x\to 1} \frac{x^2-\arcsin x}{x^3-2x-4}$.

   This can be directly evaluated: $\frac{1-\frac{\pi}{2}}{1-2-4} = \frac{\pi-2}{10}$.

   (f) **(4 points)** $\lim_{q\to-\infty} \frac{3+q^2-q^4}{7-q}$.

   Dividing both numerator and denominator by 1 to make the denominator finite, we find that

$$\lim_{q\to-\infty}\frac{3+q^2-q^4}{7-q} = \lim_{q\to-\infty}\frac{\frac{3}{q}+q-q^3}{\frac{7}{q}-1} = \lim_{q\to-\infty}\frac{0+q-q^3}{-1} = \lim_{q\to-\infty} q^3$$

   which decreases without bound as q decreases, so the limit cannot be evaluated (specifically, it “equals $-\infty$” in an idiomatic sense).