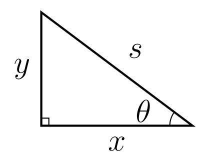
1. **(10 points)** *Find an equation of the tangent line to the curve* $y = x \arcsin x$ *at* $(\frac{1}{2}, \frac{\pi}{12})$.

   The slope of the tangent line is dictated by the derivative $\frac{dy}{dx} = \arcsin x + \frac{x}{\sqrt{1-x^2}}$ evaluated at $x = \frac{1}{2}$ to get $\arcsin \frac{1}{2} + \frac{1/2}{\sqrt{1-(1/2)^2}} = \frac{\pi}{6} + \frac{1}{\sqrt{3}}$. Then we may put this into a point-slope form to get the equation

$$y - \frac{\pi}{12} = \left(\frac{\pi}{6} + \frac{1}{\sqrt{3}}\right)\left(x - \frac{1}{2}\right).$$

2. **(18 points)** *A ten-inch long rod made of wax is firmly attached to the floor (with a swivel) eight inches from a vertical heater, which it is leaning on. The heater is melting away half an inch of wax from the free end of the rod each second.*



   Here the situation being described is depicted, with s being the length of the rod (currently 10 inches), y its height along the wall, and x its distance from the wall (which is fixed at 8 inches, so that x is actually a constant).

   (a) **(8 points)** *How quickly is the free end of the rod sliding down the heater?*

   We know from the description in the question that the length of the rod (i.e. the length s) is decreasing by 0.5 inches per second, so $\frac{ds}{dt} = -0.5$. By the Pythagorean Theorem we know that $x^2 + y^2 = s^2$, so implicitly differentiating this relationship we can solve for $\frac{dy}{dt}$:

$$\begin{aligned}
\frac{d}{dt}(x^2 + y^2) &= \frac{d}{dt}s^2 \\
\frac{dx}{dt}\frac{d}{dx}x^2 + \frac{dy}{dt}\frac{d}{dy}y^2 &= \frac{ds}{dt}\frac{d}{ds}s^2 \\
2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 2s\frac{ds}{dt} \\
2y\frac{dy}{dt} &= 2s\frac{ds}{dt} - 2x\frac{dx}{dt} \\
\frac{dy}{dt} &= \frac{s\frac{ds}{dt} - x\frac{dx}{dt}}{y}
\end{aligned}$$

   We know that $s = 10$ currently and that $x = 8$; also, since x is constant, $\frac{dx}{dt} = 0$. In addition, we can calculate $y = \sqrt{s^2 - x^2} = 6$ to determine that

$$\frac{dy}{dt} = \frac{10(-0.5) - 8 \cdot 0}{6} = \frac{-5}{6}$$

   so the end of the rod is descending the wall at five-sixths of an inch per second.

(b) **(10 points)** *How quickly is the angle between the rod and the floor changing?*

We have the same scenario and labels as above, but now we seek to find $\frac{d\theta}{dt}$. There are several trigonometric relationships we can use to describe θ, but since x and s are the quantities in the triangle about which we know the most, the relationship $\sec\theta = \frac{s}{x} = \frac{s}{8}$ is particularly fruitful. Differentiating it with respect to time:

$$\begin{aligned}
\frac{d}{dt}\sec\theta &= \frac{d}{dt}\frac{s}{8} \\
\frac{d\theta}{dt}\frac{d}{d\theta}\sec\theta &= \frac{\frac{ds}{dt}}{8} \\
\frac{d\theta}{dt}\sec\theta\tan\theta &= \frac{-0.5}{8} \\
\frac{d\theta}{dt}\frac{s}{x}\frac{y}{x} &= \frac{-1}{16} \\
\frac{d\theta}{dt} &= \frac{-x^2}{16ys} = \frac{-64}{16\cdot 60} = \frac{-1}{15}
\end{aligned}$$

Note that this result of one-fifteenth is in the unusual units of radians-per-second; if you want a more palatable unit, this is about 3.8 degrees per second.

3. **(18 points)** *The* right strophoid *is a curve satisfying the equation* $xy^2 + 5y^2 = 5x^2 - x^3$.

(a) **(14 points)** *Find a formula for* $\frac{dy}{dx}$ *on this curve.*

We implicitly differentiate both sides of the equation, and process it until only the terms x, y, and $\frac{dy}{dx}$ remain:

$$\begin{aligned}
\frac{d}{dx}\left(xy^2 + 5y^2\right) &= \frac{d}{dx}\left(5x^2 - x^3\right) \\
\frac{d}{dx}xy^2 + \frac{d}{dx}5y^2 &= 10x - 3x^2 \\
\left(\frac{d}{dx}x\right)y^2 + x\frac{d}{dx}y^2 + \frac{dy}{dx}\frac{d}{dy}5y^2 &= 10x - 3x^2 \\
1\cdot y^2 + x\frac{dy}{dx}\frac{d}{dy}y^2 + \frac{dy}{dx}(10y) &= 10x - 3x^2 \\
y^2 + 2xy\frac{dy}{dx} + 10y\frac{dy}{dx} &= 10x - 3x^2
\end{aligned}$$

And now we need to algebraically isolate $\frac{dy}{dx}$:

$$\begin{aligned}
y^2 + 2xy\frac{dy}{dx} + 10y\frac{dy}{dx} &= 10x - 3x^2 \\
(2xy + 10y)\frac{dy}{dx} &= 10x - 3x^2 - y^2 \\
\frac{dy}{dx} &= \frac{10x - 3x^2 - y^2}{2xy + 10y}
\end{aligned}$$

(b) **(4 points)** *Find the equation of the tangent line to the curve at* $(-3, 6)$.

The slope will be the value of $\frac{dy}{dx}$ at this specific point, which is

$$\left.\frac{dy}{dx}\right|_{(-3,6)} = \frac{10(-3) - 3(-3)^2 - 6^2}{2(-3)\cdot 6 + 10 \cdot 6} = \frac{-93}{24} = \frac{-31}{8}$$

So we want a line of slope $\frac{-31}{8}$ through $(-3, 6)$, which will have equation $y - 6 = \frac{-31}{8}(x+3)$.

4. **(10 points)** *For $g(t) = e^t(\sin \ln t)$, calculate $g'(t)$.*

   This is, considered on its outermost level, a product; we also see that $\sin \ln t$ will surely need a chain-rule decomposition, so we might pre-emptively let $u = \ln t$ and perform the derivative from the outside in:

$$\begin{aligned}
g'(t) &= \frac{d}{dt}\left(e^t(\sin u)\right) \\
&= \left(\frac{d}{dt}e^t\right)\sin u + e^t \frac{d}{dt}\sin u \\
&= e^t \sin u + e^t \frac{du}{dt}\frac{d}{du}\sin u \\
&= e^t \sin(\ln t) + e^t\left(\frac{d}{dt}\ln t\right)\cos u \\
&= e^t \sin(\ln t) + e^t\left(\frac{1}{t}\right)\cos(\ln t) \\
&= e^t\left(\sin(\ln t) + \frac{\cos(\ln t)}{t}\right)
\end{aligned}$$

   The last line is a purely cosmetic simplification and is unnecessary.

5. **(12 points)** *Find $\frac{d}{dx}\arctan\sqrt{e^x}$.*

   This is a chain of several embedded simple functions. If we give them all names, we might let $v = e^x$ and $u = \sqrt{v} = \sqrt{e^x}$, so that

$$\begin{aligned}
\frac{d}{dx}\arctan\sqrt{e^x} &= \frac{d}{dx}\arctan u \\
&= \frac{du}{dx}\frac{d}{du}\arctan u \\
&= \left(\frac{d}{dx}\sqrt{v}\right)\frac{d}{du}\arctan u \\
&= \frac{dv}{dx}\left(\frac{d}{dv}\sqrt{v}\right)\frac{d}{du}\arctan u \\
&= \left(\frac{d}{dx}e^x\right)\left(\frac{d}{dv}\sqrt{v}\right)\frac{d}{du}\arctan u \\
&= e^x \frac{1}{2\sqrt{v}}\frac{1}{u^2+1} \\
&= e^x \frac{1}{2\sqrt{e^x}}\frac{1}{\sqrt{e^x}^2+1} = \frac{\sqrt{e^x}}{2e^x+2}
\end{aligned}$$

   The last form is a simplification and is not necessary.

6. **(8 points)** *Estimate the following values using appropriate linear approximations.*

   (a) **(4 points)** 1.007^6.

   We observe that this number is very close to 1, which has an easily calculated sixth power. We thus use a linear approximation to the function $f(x) = x^6$ near the point $a = 1$. Since $f'(x) = 6x^5$, we ascertain that $f(1) = 1$ and $f'(1) = 6$ and so

   $$f(1 + 0.007) \approx f(1) + 0.007f'(1) = 1 + 0.042 = 1.042$$

   Note that the actual value of 1.007^6 is approximately 1.0427419, so this is a good but not great approximation.

   (b) **(4 points)** $\sqrt[3]{-26.9973}$.

   We observe that this number is very close to -27, which has a nice cube root. We thus use a linear approximation to the function $f(x) = \sqrt[3]{x}$ near the point $a = -27$. Since $f'(x) = \frac{1}{3x^{2/3}}$, we ascertain that $f(-27) = -3$ and $f'(-27) = \frac{1}{3 \cdot 9} = \frac{1}{27}$ and so

   $$f(-27 + 0.0027) \approx f(-27) + 0.0027f'(27) = -3 + \frac{0.0027}{27} = -2.9999$$

   Note that the actual value of $\sqrt{-26.9973}$ is approximately 2.999899996, rendering this result accurate to 8 decimal places!

7. **(12 points)** *Find the maximum and minimum values of the function* $f(x) = \frac{x^2+2x+10}{x+1}$ *on the interval* $[0, 5]$.

   $f'(x) = \frac{(x+1)(2x+2)-(x^2+2x+10)}{(x+1)^2} = \frac{x^2+2x-8}{(x+1)^2}$, which exists everywhere we are interested in (its asymptote at $x = -1$ is fortunately outside of the interval $[0, 5]$), so our only relevant critical points are where $x^2 + 2x - 8 = 0$, which occurs when $x = -4$ or $x = 2$. Note that -4 is not in the interval $[0, 5]$, so our candidates for maximum are thus 0, 2, and 5. Then $f(0) = \frac{10}{1} = 10$, $f(2) = \frac{18}{3} = 6$, and $f(5) = \frac{45}{6} = 7.5$, so $(0, 10)$ is the maximum and $(5, 7.5)$ is the minimum.

8. **(12 points)** *For* $y = \sqrt{\frac{e^x+2}{\ln x} + 1}$, *calculate* $\frac{dy}{dx}$.

   This expression is a chain on its outermost level with $u = \frac{e^x+2}{\ln x} + 1$, and we expect that in differentiating u, we will also need the quotient rule. The full derivation follows:

   $$\begin{aligned}
   \frac{dy}{dx} &= \frac{d}{dx}\sqrt{u} \\
   &= \frac{du}{dx}\frac{d}{du}\sqrt{u} \\
   &= \frac{d}{dx}\left(\frac{e^x+2}{\ln x} + 1\right)\frac{1}{2\sqrt{u}} \\
   &= \frac{(\ln x)\frac{d}{dx}(e^x+2) - (e^x+2)\frac{d}{dx}\ln x}{(\ln x)^2} \frac{1}{2\sqrt{\frac{e^x+2}{\ln x} + 1}} \\
   &= \frac{(\ln x)e^x - (e^x+2)\frac{1}{x}}{2(\ln x)^2\sqrt{\frac{e^x+2}{\ln x} + 1}}
   \end{aligned}$$