1. **(17 points)** Answer the following questions:

   (a) **(8 points)** *Find the general antiderivative of* $f(x) = 2\cos x - \frac{3+5x}{x^2} + \frac{4}{x^2+1} - 20$.
   The second term requires some cleanup; we may rewrite this expression as

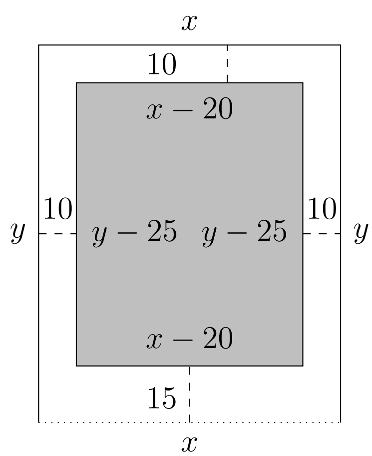
   $$f(x) = 2\cos x - 3x^{-2} + 5x^{-1} + \frac{4}{x^2+1} - 20$$

   and then antidifferentiate term=by=term to find that

   $$F(x) = 2\sin x + 3x^{-1} + 5\ln|x| + 4\arctan x - 20x + C$$

   (b) **(9 points)** *Given that* $h'(x) = 3x^5 - 7x + 2 + \frac{6}{x}$ *and* $h(1) = 5$, *find a formula for* $h(x)$.
   Using a general antiderviative, we know that $h(x) = \frac{1}{2}x^6 - \frac{7}{2}x^2 + 2x + 6\ln|x| + C$. Then $h(1) = \frac{1}{2} - \frac{7}{2} + 2 + 6\ln 0 + C$ and since we want $h(1)$ to be 5, we thus have $5 = -1 + C$ so $C = 6$, giving the formula $h(x) = \frac{1}{2}x^6 - \frac{7}{2}x^2 + 2x + 6\ln|x| + 6$.

2. **(24 points)** *An industrial manufacturer has budgeted 8000 square feet of floor space for a rectangular factory with a loading dock running along one side. They want to fill it as full as possible with machinery, but regulations require 10 feet of open floor space along each wall, and 15 feet of open floor space along the loading dock. What dimensions for the factory will maximize the quantity of* usable *floor space (places they can legally put machines)?*



The above drawing is a representation of the scenario described; we assign the two dimensions of the factory the labels of x and y (we could alternatively label the dimensions of the usable area with x and y, which would give correct results, but it would make the arithmetic a bit messier). Since there is a gap of 10 feet along each wall and 15 feet along the loading dock, the height of the usable area will be 25 feet less than the area of the factory; likewise, the width of the usable space is 20 feet less than the width of the factory, so the usable region is an $(x-20) \times (y-25)$ rectangle.

Our constraint is that the factory as a whole has an area of 8000 square feet, so we are constrained that the non-negative quantities x and y must be such that $xy = 8000$. What we seek to maximize is the usable area $(x-20)(y-25)$; rephrasing the above constraint as $y = \frac{8000}{x}$, we see that the area is

$$A(x) = (x-20)\left(\frac{8000}{x} - 25\right) = 8000 - 25x - \frac{160000}{x} + 500 = 850 - 25x - \frac{160000}{x}$$

Our limits on x are dictated by the need for x to be at least 20 to permit the wall clearance, and for y to be at least 25 to permit wall-and-dock clearance. Thus $x \geq 20$ and $\frac{8000}{x} \geq 25$, so $20 \leq x \leq 320$.

Solving this maximization problem, we observe that $A'(x) = -25 + \frac{160000}{x^2}$. This is undefined when $x = 0$, and is zero when $x^2 = 6400$, or when $x = \pm 80$. We note that $x = 0$ and $x = -80$ are outside our interval, leaving us with the potential optima $x = 20$, $x = 80$, and $x = 320$. Unsurprisingly, $A(20) = 0$ and $A(320) = 0$, since both of these are trivial factories which are either too narrow or too short to contain any machines, and we are left with an 80×100 factory as optimal (which has a 60×75 floor space suitable for placing 4500 square feet of machinery).

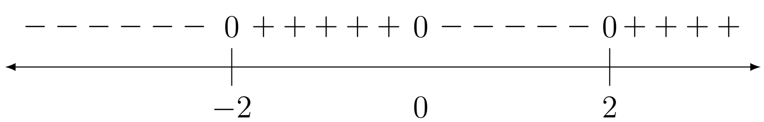
3. **(24 points)** *Answer the following questions related to the shape of the graph of the function* $f(x) = x^4 - 8x^2 + 8$.

   (a) **(4 points)** *What are $f(x)$'s long term behaviors as x grows very large and as x grows very negative? Describe each direction in either words or symbols.*

   Since the dominant term in $f(x)$ is x^4, we know that $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} x^4$, so that as x gets very large in magnitude in *either* direction, $f(x)$ becomes very large (and positive).

   (b) **(6 points)** *Where is $f(x)$ increasing? Where is it decreasing? Label which is which.*

   This answer is dependent on the sign of $f'(x)$ at various values. We know that $f'(x) = 4x^3 - 16x = 4x(x-2)(x+2)$, which is zero when $x = -2$, $x = 0$, or $x = 2$. We can probe at $x = -3$, $x = -1$, $x = 1$, and $x = 3$ to look at the regions around the zeroes: note that $f'(-3) = -60$, $f'(-1) = 12$, $f'(1) = -12$, and $f'(3) = 60$, so the sign of $f'(x)$ behaves as such:



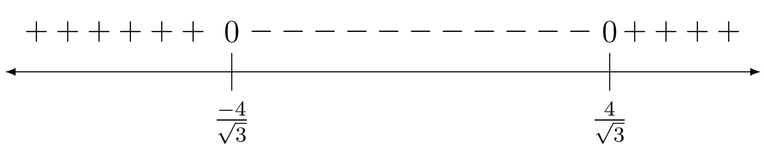
   Thus $f(x)$ is increasing when $-2 < x < 0$ or $x > 2$ and is decreasing when $x < -2$ or $0 < x < 2$.

   (c) **(6 points)** *What are its critical points, and is each a local maximum, a local minimum, or neither?*

   As seen above, this function achieves a zero derivative, and thus criticality, at $x = -2$, $x = 0$ and $x = 2$. The critical points at $x = -2$ and $x = 2$ are transition points from decrease to increase which is thus a local maximum. The critical point at $x = 0$ is a transition point from increase to decrease which is thus a local maximum.

   (d) **(8 points)** *Where is it concave up? Where is it concave down? Label which is which. Where, if anywhere, are its points of inflection?*

   This answer is dependent on the sign of $f''(x)$ at various values. We know that $f''(x) = 12x^2 - 16$, which is zero when $x = \pm\frac{4}{\sqrt{3}}$. We can probe at $x = -2$, $x = 0$, and $x = 2$ to look at the regions around the zeroes: note that $f''(\pm 2) = 32$ and $f''(0) = -16$, so the sign of $f''(x)$ behaves as such:



Thus $f(x)$ is concave up when $x < \frac{-4}{\sqrt{3}}$ or $x > \frac{4}{\sqrt{3}}$, and concave down when $\frac{-4}{\sqrt{3}} < x < \frac{4}{\sqrt{3}}$, with points of inflection at $x = \frac{-4}{\sqrt{3}}$ and $x = \frac{4}{\sqrt{3}}$.

4. **(12 points)** *Answer the following questions.*

   (a) **(5 points)** *A car's speed in feet per second measured over a 30-second period is given in the table below:*

Time in seconds	*0*	*5*	*10*	*15*	*20*	*25*	*30*
Speed in ft/sec	*25*	*31*	*35*	*43*	*47*	*45*	*41*

   Use one of the two standard techniques to estimate the distance traveled in this time.
   Using a left-side Riemann sum, the total distance traveled is approximated by

$$5 \times 25 + 5 \times 31 + 5 \times 35 + 5 \times 43 + 5 \times 47 + 5 \times 45 = 5 \times 226 = 1130$$

   or with a right-side Riemann sum,

$$5 \times 31 + 5 \times 35 + 5 \times 43 + 5 \times 47 + 5 \times 45 + 5 \times 41 = 5 \times 242 = 1210$$

   with the truth likely somewhere between these two results.

   (b) **(7 points)** *Simplify* $\frac{d}{dx}\int_x^{e^x} \frac{t^3-3t}{\arctan t+4}dt$.
   Let $f(t) = \frac{t^3-3t}{\arctan t+4}$, and let $F(t)$ be its (not symbolically calculatable) antiderivative. Then, by the Fundamental Theorem of Calculus,

$$\int_x^{e^x} \frac{t^3-3t}{\arctan t+4}dt = \int_x^{e^x} d(t)dt = F(e^x) - F(x)$$

   and since we want the derivative of this expression, we may use the chain rule to find that

$$\frac{d}{dx}\left(F(e^x) - F(x)\right) = e^x F'(e^x) - F'(x) = e^x f(e^x) - f(x) = e^x \frac{e^{3x} - 3e^x}{\arctan e^x + 4} - \frac{x^3 - 3x}{\arctan x + 4}$$

5. **(23 points)** *Evaluate the following limits; if they cannot be evaluated, show why not.*

   (a) $\lim\limits_{x\to\infty} \frac{e^{x/10}}{x^3}$.
   This is an $\frac{\infty}{\infty}$ indeterminate form and remains one for several applications of L'Hôpital's rule (with the chain rule used in the numerator:

$$\lim_{x\to\infty} \frac{e^{x/10}}{x^3} = \lim_{x\to\infty} \frac{\frac{1}{10}e^{x/10}}{3x^2} = \lim_{x\to\infty} \frac{\frac{1}{100}e^{x/10}}{6x} = \lim_{x\to\infty} \frac{\frac{1}{1000}e^{x/10}}{6} = \lim_{x\to\infty} \frac{e^{x/10}}{6000}$$

   whose numerator continues to grow without bound, so the limit is nonexistent (and is specifically describable as $+\infty$).

   (b) $\lim\limits_{\theta\to 0} \frac{\cos\theta}{2-\sin\theta}$.
   This limit is not an intederminate form, and directly evaluates to $\frac{1}{2-0} = \frac{1}{2}$.

   (c) $\lim\limits_{s\to 0} \frac{e^s - e^{-s} - 2s}{s^3}$.
   This limit is a $\frac{0}{0}$ indeterminate form and remains one through several applications of L'Hôpital's rule:

$$\lim_{s\to 0} \frac{e^s - e^{-s} - 2s}{s^3} = \lim_{s\to 0} \frac{e^s + e^{-s} - 2}{3s^2} = \lim_{s\to 0} \frac{e^s - e^{-s}}{6s} = \lim_{s\to 0} \frac{e^s + e^{-s}}{6}.$$

   This last expression is evaluatable as $\frac{1+1}{6} = \frac{1}{3}$.

(d) $\lim_{y\to 0} \frac{\cos(7y)-\cos(3y)}{y^2}$.

This limit is a $\frac{0}{0}$ indeterminate form, so we apply L'Hôpital's rule:

$$\lim_{y\to 0} \frac{\cos(7y)-\cos(3y)}{y^2} = \lim_{y\to 0} \frac{-7\sin(7y)+3\sin(3y)}{2y}$$

which is still a $\frac{0}{0}$ form, so we use L'Hôpital's rule again:

$$\lim_{y\to 0} \frac{-7\sin(7y)+3\sin(3y)}{2y} = \lim_{y\to 0} \frac{-49\cos(7y)+9\cos(3y)}{2} = \frac{-49+9}{2} = -20.$$

(e) $\lim_{x\to\pi/2^-} (\cos x)(\sec 5x)$.

This limit would appear to be a $\infty\cdot 0$ indeterminate form on direct evaluation, so it must be recast as a fraction. The most straightforward way to do so is to rewrite it as $\lim_{x\to\pi/2+} \frac{\cos x}{\cos 5x}$, which is a $\frac{0}{0}$ indeterminate form. Using L'Hôpital's rule,

$$\lim_{x\to\pi/2+} \frac{\cos x}{\cos 5x} = \lim_{x\to\pi/2+} \frac{-\sin x}{-5\sin 5x} = \frac{-1}{-5} = \frac{1}{5}.$$