1. **(7 points)** *Answer the following questions relevant to the limit* $\lim_{x\to-3} 4x+5=-7$.

   (a) **(2 points)** *In terms of the parameter* ε*, what formal condition corresponds to the goal that* $4x+5$ *be "very close" to* -7*?*

   The distance from $4x+5$ to -7 is $|(4x+5)-(-7)| = |4x+12|$. Making this distance "very small" is, in a formal sense, bounding it above by the (presumably very small) positive parameter ε, so the condition is that $|4x+12| < \varepsilon$. Variant phrasings of this same condition might be that $-\varepsilon < 4x+12 < \varepsilon$ or that $-7-\varepsilon < 4x+5 < -7+\varepsilon$.

   (b) **(2 points)** *In terms of the parameter* δ*, what formal condition corresponds to the restriction that* x *is "very close" to* -3*?*

   The distance from x to -3 is $|x-(-3)| = |x+3|$. Making this distance "very small" (but nonzero!) is, in a formal sense, making it a positive value smaller than the (presumably very small) parameter δ, so the condition is that $0 < |x+3| < \delta$. Variant phrasings of this same condition might be that $-\delta < x-3 < \delta$ but $x-3 \neq 0$; or that $3-\delta < x < 3+\delta$ but $x \neq 3$.

   (c) **(3 points)** *When* $\varepsilon = 0.2$*, what choice of* δ *suffices to demonstrate that the limit is true? Show your work.*

   As seen above, this is equivalent to the requirement that $4x+5$ is within a distance 0.2 of -7, or in other words that $-7.2 < 4x+5 < -6.8$. Solving for x in this inequality, we can see that this condition is met when $-3.05 < x < -2.95$. Thus, in order to get $4x+5$ within this distance of -7, we must choose an x within the distance 0.05 of -3, i.e., we need to choose $\delta = 0.05$ (or smaller).

2. **(6 points)** *Find values of* a *and* b *such that the following piecewise function is continuous everywhere.*

$$f(x) = \begin{cases} ax^2 & \text{if } x < 5 \\ \dfrac{x+3}{x-4} & \text{if } 5 \leq x \leq 11 \\ 2x+b & \text{if } x > 11 \end{cases}$$

Regardless of what values of a and b are chosen, the function given will surely be continuous everywhere except at $x=5$ and $x=11$; note that ax^2 and $2x+b$ are continuous everywhere, while $\frac{x+3}{x-4}$ is continuous everywhere except at $x=4$, which is conveniently outside the interval where this function is used. Thus, we must choose a and b so as to repair those potential discontinuities at $x=5$ and $x=11$.

At $x=5$, continuity would require that $f(5)$, $\lim_{x\to5^-} f(x)$, and $\lim_{x\to5^+} f(x)$ all be equal. Note that $\lim_{x\to5^+} f(x) = f(5) = \frac{5+3}{5-4} = 8$, but that $\lim_{x\to5^-} f(x) = a\cdot5^2 = 25a$. So we wish for $25a$ to equal 8, which we can accomplish by making $a = \frac{8}{25}$.

At $x=11$, continuity would require that $f(11)$, $\lim_{x\to11^-} f(x)$, and $\lim_{x\to11^+} f(x)$ all be equal. Note that $\lim_{x\to11^-} f(x) = f(11) = \frac{11+3}{11-1} = 2$, but that $\lim_{x\to11^+} f(x) = 2\cdot11+b = 22+b$. So we wish for $22+b$ to equal 2, which we can accomplish by making $b=-20$.

3. **(7 points)** *Calculate the following limits at infinity; when possible, even if the limit does not exist, use one of the "limit idioms" to describe its behavior.*

(a) **(2 points)** $\lim_{x\to-\infty} 3+2x^2-x^3$.

The dominant term in this polynomial is $-x^3$; that has very large magnitude as x's own magnitude increases, so the limit doesn't exist. However, we note that it does so in an infinite manner for which we have idiomatic phrasings: when x is a large-magnitude negative number, x^3 remains negative and has even greater magnitude, which when negated will be a large-magnitude positive number. Thus $\lim_{x\to-\infty} 3+2x^2-x^3 = +\infty$.

(b) **(2 points)** $\lim_{s\to+\infty} \frac{s^2-3s+2}{6-s^2}$.

The dominant terms in the numerator and denominator are s^2 and $-s^2$ respectively; thus, for s very large, $\frac{s^2-3s+2}{6-s^2} \approx \frac{s^2}{-s^2}$ and so $\lim_{s\to+\infty} \frac{s^2-3s+2}{6-s^2} = \lim_{s\to+\infty} \frac{s^2}{-s^2} = -1$.

An alternative approach, using the reduction-to-zero technique rather than the dominant-term inspection, follows:

$$\lim_{s\to+\infty} \frac{s^2-3s+2}{6-s^2} = \lim_{s\to+\infty} \frac{\frac{s^2-3s+2}{s^2}}{\frac{6-s^2}{s^2}} = \frac{\lim_{s\to+\infty} 1-\frac{3}{s}+\frac{2}{s^2}}{\lim_{s\to+\infty} \frac{6}{s^2}-1} = \frac{1-0+0}{0-1} = -1.$$

(c) **(3 points)** $\lim_{u\to-\infty} \left(2\arctan u - \frac{2u^3-7u}{u^5-u^4+1}\right)$.

First note that $\lim_{u\to-\infty} \left(2\arctan u - \frac{2u^3-7u}{u^5-u^4+1}\right) = 2\left(\lim_{u\to-\infty} \arctan u\right) - \left(\lim_{u\to-\infty} \frac{2u^3-7u}{u^5-u^4+1}\right)$. The former limit is a known form: $\lim_{u\to-\infty} \arctan u = \frac{-\pi}{2}$. For the latter, we may use either a dominant-term inspection or a reduction to zero. We might note that $\frac{2u^3-7u}{u^5-u^4+1} \approx \frac{2u^3}{u^5}$ and so

$$\lim_{u\to-\infty} \frac{2u^3-7u}{u^5-u^4+1} = \lim_{u\to-\infty} \frac{2u^3}{u^5} = \lim_{u\to-\infty} \frac{2}{u^2} = 0$$

or alternatively

$$\lim_{u\to-\infty} \frac{2u^3-7u}{u^5-u^4+1} = \lim_{u\to-\infty} \frac{\frac{2u^3-7u}{u^5}}{\frac{u^5-u^4+1}{u^5}} = \frac{\lim_{u\to-\infty} \frac{2}{u^2}-\frac{7}{u^4}}{\lim_{u\to-\infty} 1-\frac{1}{u}+\frac{1}{u^5}} = \frac{0-0}{1-0+0} = 0.$$

In either case, however, we will get the eventual result that

$$\lim_{u\to-\infty} \left(2\arctan u - \frac{2u^3-7u}{u^5-u^4+1}\right) = 2\left(\frac{-\pi}{2}\right) - 0 = -\pi.$$