1. **(5 points)** *Find a value k such that the matrix* $\begin{pmatrix} 1 & 2 & 4 \\ 5 & k & 0 \\ -3 & -2 & 1 \end{pmatrix}$ *is singular.*

   A matrix is singular if and only if its determinant is zero, so we seek a value k such that

   $$\begin{vmatrix} 1 & 2 & 4 \\ 5 & k & 0 \\ -3 & -2 & 1 \end{vmatrix} = 0$$

   which will be the case when

   $$k + 0 - 40 - 0 - 10 + 12k = 0$$

   whose solution is $k = \frac{50}{13}$.

   This problem could also be solved (with more difficulty) by row-reducing the matrix to get something akin to $\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & \frac{13}{4} \\ 0 & 0 & \frac{50-13k}{4} \end{pmatrix}$, which fails to have a pivot in the last row when $\frac{50-13k}{4} = 0$, which is true when $k = \frac{50}{13}$.

2. **(10 points)** *For each of the following subsets S of a named vector space V, explain whether S is or is not a subspace of V and why.*

   (a) $V = \mathbb{R}^{2\times 2}$, $S = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a + d = 0\}$.
   Here S is a subspace. Two elements of S would have the form $\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} d & e \\ f & -d \end{smallmatrix}\right)$; their sum would have the form $\left(\begin{smallmatrix} a+d & b+e \\ c+f & -a-d \end{smallmatrix}\right)$, which still meets the criterion for membership in S since $(a + d) + (-a - d) = 0$; likewise a constant multiple of the former would be $\left(\begin{smallmatrix} ka & kb \\ kc & -ka \end{smallmatrix}\right)$, which also is a member of S since $ka + (-ka) = 0$.

   (b) $V = \mathbb{R}^3$, $S = \{(x, 3x, 4x + 2)^T : x \in \mathbb{R}\}$.
   In this case S is not a subspace, since for example $\left(\begin{smallmatrix} 0 \\ 0 \\ 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 \\ 3 \\ 14 \end{smallmatrix}\right)$ are in S, but neither their sum $\left(\begin{smallmatrix} 1 \\ 3 \\ 16 \end{smallmatrix}\right)$ nor the scalar multiple of the first $\left(\begin{smallmatrix} 0 \\ 0 \\ 8 \end{smallmatrix}\right)$ is in S.

   (c) $V = P_2$, $S = \{f(x) : f(2) \leq 0\}$.
   In this example S is not a subspace, although it is closed under addition: if $f(2) \leq 0$ and $g(2) \leq 0$, it is certainly the case that $f(x) + g(x) \leq 0$. It fails to satisfy the scalar multiplication principle, since if $f(2) < 0$, then $-f(2) > 0$ and thus $-1f$ is not an element of S. More concretely, we might consider the polynomial $x^2 - 5$, which is in S, and note that its scalar multiple $5 - x^2$ is not in S.

3. **(10 points)** *Let* $A = \begin{pmatrix} 4 & 1 & 6 \\ 2 & 3 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 3 & 0 \\ -2 & 2 & -4 \end{pmatrix}$, *and* $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$. *For each of the following arithmetic expressions, either calculate its value or explain briefly why it cannot be calculated.*

   (a) $(A + B)\mathbf{v}$.

   $$\left[\begin{pmatrix} 4 & 1 & 6 \\ 2 & 3 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 0 \\ -2 & 2 & -4 \end{pmatrix}\right] \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 & 4 & 6 \\ 0 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \cdot 1 + 4 \cdot 0 - 6 \cdot 2 \\ 0 \cdot 1 + 5 \cdot 0 - 1 \cdot 2 \end{pmatrix} = \begin{pmatrix} -7 \\ -2 \end{pmatrix}$$

(b) AB.

Since A has 3 columns and B has 2 rows, this product is not valid.

(c) AB^T.

$$\begin{pmatrix} 4 & 1 & 6 \\ 2 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 3 & 2 \\ 0 & -4 \end{pmatrix} = \begin{pmatrix} 4\cdot 1 + 1\cdot 3 + 6\cdot 0 & 4\cdot -2 + 1\cdot 2 + 6\cdot -4 \\ 2\cdot 1 + 3\cdot 3 + 5\cdot 0 & 2\cdot -2 + 3\cdot 2 + 5\cdot -4 \end{pmatrix} = \begin{pmatrix} 7 & -30 \\ 11 & -18 \end{pmatrix}$$

(d) $A^T B\mathbf{v}$.

$$\begin{pmatrix} 4 & 2 \\ 1 & 3 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ -2 & 2 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} 1\cdot 1 + 3\cdot 0 - 0\cdot 2 \\ -2\cdot 1 + 2\cdot 0 + 4\cdot 2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 16 \\ 19 \\ 36 \end{pmatrix}$$

4. **(15 points)** *Let* $A = \begin{pmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{pmatrix}$.

(a) **(10 points)** *Calculate the inverse of A.*

Plausible approaches to this calculation include converting the augmented matrix $\left(\begin{array}{c|c} A & I \end{array} \right)$ to $\left(\begin{array}{c|c} I & A^{-1} \end{array} \right)$ with elementary row operations, or using the adjoint formula for a matrix inverse. The former approach is shown first, using some nonstandard choices of elementary row operations to avoid unpleasant intermediary fractions:

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array} \right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -1 & -2 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array} \right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -1 & -2 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & 0 & 3 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{array} \right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 2 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 3 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{array} \right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{array} \right)$$

or alternatively, using the adjoint formula:

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \frac{1}{\begin{vmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{vmatrix}} \begin{pmatrix} \begin{vmatrix} -2 & 0 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 4 & 3 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 4 & 3 \\ -2 & 0 \end{vmatrix} \\ -\begin{vmatrix} -1 & 0 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} \\ \begin{vmatrix} -1 & -2 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ -1 & -2 \end{vmatrix} \end{pmatrix} = \frac{1}{12} \begin{pmatrix} -6 & -6 & 6 \\ 3 & -3 & -3 \\ 2 & 6 & 2 \end{pmatrix}$$

(b) **(5 points)** *Find a solution* $\mathbf{v}$ *to the matrix equation*

$$\begin{pmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 12 \\ -12 \\ 8 \end{pmatrix}$$

This can of course be solved by row-reducing the augmented matrix, but armed with the inverse from above, it is much easier to observe that since $A\mathbf{v} = \left(\begin{smallmatrix} 12 \\ -12 \\ 8 \end{smallmatrix}\right)$, it is the case that $\mathbf{v} = A^{-1}\left(\begin{smallmatrix} 12 \\ -12 \\ 2 \end{smallmatrix}\right)$ and so:

$$\mathbf{v} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 12 \\ -12 \\ 8 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ -\frac{8}{3} \end{pmatrix}$$

5. **(25 points)** *Answer the following questions.*

(a) **(10 points)** *Calculate the determinant* $\begin{vmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{vmatrix}$.

Since Type III row operations don't change the determinant, we can make this question massively simpler by adding the last row to the first (or vice versa):

$$\begin{vmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 5 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{vmatrix}$$

and then use a cofactor expansion along the first row:

$$\begin{vmatrix} 0 & 0 & 0 & 5 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{vmatrix} = 0 - 0 + 0 - 5 \cdot \begin{vmatrix} 0 & 3 & 1 \\ 0 & 0 & 2 \\ -1 & -1 & -1 \end{vmatrix} = -5 \cdot \left(-0 + 0 - 2 \begin{vmatrix} 0 & 3 \\ -1 & -1 \end{vmatrix} \right) = 30$$

Other approaches such as reducing the matrix to a triangular form (which could be done with a single Type III move!) and multiplying on the diagonal would also work.

(b) **(7 points)** *What is the entry in the first row and second column of* $\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{pmatrix}^{-1}$*?*

Using the adjoint formula for an inverse, we know that it is

$$\frac{-\begin{vmatrix} 1 & 1 & 3 \\ 0 & 2 & 2 \\ -1 & -1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{vmatrix}} = -\frac{4 - 2 + 0 + 2 + 0 + 6}{30} = \frac{-1}{3}.$$

(c) **(8 points)** Calculate the value of y in the following equation:

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 2 \end{pmatrix}$$

Cramer's rule tells us that y has a value given by the ratio of determinants

$$\frac{\begin{vmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 2 \\ -1 & -1 & 2 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{vmatrix}}$$

The denominator is known to be 30 from the previous result. The numerator can be calculated using a cofactor expansion on the third row:

$$\begin{vmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 2 \\ -1 & -1 & 2 & 2 \end{vmatrix} = 0 - 0 + 0 - 2 \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ -1 & -1 & 2 \end{vmatrix} = -2(6 - 3 + 0 + 3 + 0 + 3) = -18$$

so $y = \frac{-18}{30} = \frac{-3}{5}$. Incidentally, $w = \frac{-4}{5}$, $x = \frac{3}{5}$, and $z = \frac{3}{5}$ (although those were not requested in the problem).

6. **(35 points)** Answer the following related questions:

(a) **(15 points)** *For the following system of equations, determine its solution set or describe it as inconsistent:*

$$\begin{cases} x_1 - x_2 + 3x_3 + 2x_4 = 1 \\ -x_1 + x_2 - 2x_3 + x_4 = -2 \\ 2x_1 - 2x_2 + 7x_3 + 7x_4 = 1 \end{cases}$$

Performing Gauss-Jordan elimination on the associated augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & -1 & 3 & 2 & 1 \\ -1 & 1 & -2 & 1 & -2 \\ 2 & -2 & 7 & 7 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -1 & 3 & 2 & 1 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 3 & -1 \end{array}\right)$$
$$\sim \left(\begin{array}{cccc|c} 1 & -1 & 0 & -7 & 4 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

From the above we can conclude that $x_1 - x_2 - 7x_4 = 4$ and $x_3 + 3x_4 = -1$. The obvious way to recast this as dependent on a single variable is to rephrase x_1 and x_3 in terms of x_2 and x_4; thus we have the solution

$$(x_1, x_2, x_3, x_4) = (4 + 7x_4 + x_2, x_2, -1 - 3x_4, x_4)$$

where x_2 and x_4 could be any real numbers.

(b) **(10 points)** *Calculate the dimension of the nullspace of the matrix* $\begin{pmatrix} 1 & -1 & 3 & 2 \\ -1 & 1 & -2 & 1 \\ 2 & -2 & 7 & 7 \end{pmatrix}$ *, and find a basis.*

As seen above,

$$\begin{pmatrix} 1 & -1 & 3 & 2 \\ -1 & 1 & -2 & 1 \\ 2 & -2 & 7 & 7 \end{pmatrix} \text{ is row-equivalent to } \begin{pmatrix} 1 & -1 & 0 & -7 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and since the above matrix has two pivotless columns, there are two free variables in a nullvector of this matrix and the nullspace has dimension 2. Specifically, elements of the nullspace have the form $(7x_4 + x_2, x_2, -3x_4, x_4)^T$, which is spanned by the basis set

$$\left\{ \begin{pmatrix} 7 \\ 0 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

(c) **(5 points)** *Is the set of vectors* $\left\{ \begin{pmatrix} 1 \\ -1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 7 \\ 7 \end{pmatrix} \right\}$ *linearly independent? Explain your reasoning.*

Since the Gauss-Jordan process on a matrix with these vectors as rows eventually produced a zero row, we know that these vectors are not linearly independent. Specifically, if we look back at our Gauss-Jordan process, we might pull out of it the fact that

$$\begin{pmatrix} -1 \\ 1 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 7 \\ 7 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ -1 \\ 3 \\ 2 \end{pmatrix}$$

or in other words that

$$3 \begin{pmatrix} 1 \\ -1 \\ 3 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 7 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(d) **(5 points)** *Is the set of vectors* $\left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix} \right\}$ *linearly independent? Explain your reasoning.*

This set of vectors is not linearly independent. There are several possible arguments to be made to show this point:

- The nontrivial linear combination $1 \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} + 0 \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix}$ is the zero vector.
- The row-reduction of a matrix with these vectors as the columns has a column which lacks a pivot.
- These vectors are all elements of the 3-dimensional vector space $\mathbb{R}^3$, so the dimension of the space spanned by these four vectors is at most 3; this 4-element set thus cannot be linearly independent.