1. **(15 points)** *The matrix* $A = \begin{pmatrix} 2 & 2 & 4 & 7 \\ 1 & 6 & 12 & 6 \\ -2 & 2 & 4 & -5 \end{pmatrix}$ *is row-equivalent to* $\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. *Using this information, determine a basis for each of the following spaces.*

   (a) **(5 points)** $N(A)$.

   We see from the row-reduction above that if $A\mathbf{x} = \mathbf{0}$, then $x_1 = -3x_4$, $2x_2 = -4x_3 - x_4$. If we let $x_3 = k$ and $x_4 = 2\ell$, then $\mathbf{x} = \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} k + \begin{pmatrix} -6 \\ -1 \\ 0 \\ 2 \end{pmatrix} \ell$, so $N(A)$ has basis set

   $$\left\{ \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ -1 \\ 0 \\ 2 \end{pmatrix} \right\}.$$

   (b) **(5 points)** $R(A)$.

   From the row-reduction above, A has rank 2 so $R(A)$ is a two-dimensional subspace of $\mathbb{R}^3$. There are multiple possible answers but the most straightforward is to select the first two columns (each of which has a pivot in it after row-reduction) of A to get $\left\{ \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} \right\}$. Note that the third and fourth columns can be given as linear combinations of these two: $\begin{pmatrix} 4 \\ 12 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 6 \\ -5 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix}$. Since they are not linearly independent from the vectors selected, they should not be included in the basis.

   (c) **(5 points)** $R(A^T)$.

   The nonzero rows of the row-reduction, rearranged as column vectors, will suffice. This approach yields the basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 4 \\ 1 \end{pmatrix} \right\}$.

2. **(15 points)** *Find a least-squares solution to the following inconsistent system of equations:*

$$\begin{cases} x + y = 6 \\ x - y = 1 \\ \quad 2y = 9 \end{cases}$$

The system described here can alternatively be presented as the matrix equation

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 9 \end{pmatrix}$$

which we project onto a space where it is solvable by multiplying the left and right sides of the equation by the transpose of the coefficient matrix:

$$\begin{pmatrix}1 & 1 & 0\\ 1 & -1 & 2\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\\ 0 & 2\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} = \begin{pmatrix}1 & 1 & 0\\ 1 & -1 & 2\end{pmatrix}\begin{pmatrix}6\\ 1\\ 9\end{pmatrix}$$

$$\begin{pmatrix}2 & 0\\ 0 & 6\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} = \begin{pmatrix}7\\ 23\end{pmatrix}$$

So $2\hat{x} = 7$ and $6\hat{y} = 23$, which easily yields the solution $(\hat{x}, \hat{y}) = (\frac{7}{2}, \frac{23}{6})$.

3. **(15 points)** *For each of the following maps L on a vector space V, determine with explanation whether L is a linear transformation.*

   (a) $L(f) = e^{f(x)}$ *on $V = C(\mathbb{R})$, the space of continuous functions on the real number line.*
   This is not a linear transformation, since, for example, $2L(x) = 2e^x$ and $L(2x) = e^{2x}$, which are different results.

   (b) $L((x, y)^{\mathrm{T}}) = (x^2 - y^2, x^2 + y^2)^{\mathrm{T}}$ *on $V = \mathbb{R}^2$.*
   This is not a linear transformation, since, for example, $L(2(1, 0)^{\mathrm{T}}) = L((2, 0)^{\mathrm{T}}) = (4, 4)$, while $2L((1, 0)^{\mathrm{T}}) = 2(1, 1) = (2, 2)$.

   (c) $L(f) = (x + 1)f''(x)$ *on $V = P_3$.*
   This is a linear transformation. For any polynomials f and g, and real number k, it will be true that:

$$L(kf) = (x+1)\frac{d^2}{dx^2}kf(x) = k(x+1)f''(x) = kL(f),$$

   and

$$L(f+g) = (x+1)\frac{d^2}{dx^2}(f(x)+g(x)) = (x+1)(f''(x)+g''(x)) = (x+1)f''(x)+(x+1)g''(x) = L(f)+L(g)$$

   One might alternatively note that, on the standard basis $[1, x, x^2, x^3]$ for P_3, this has matrix representation $\begin{pmatrix}0 & 0 & 2 & 0\\ 0 & 0 & 2 & 6\\ 0 & 0 & 0 & 6\\ 0 & 0 & 0 & 0\end{pmatrix}$.

4. **(15 points)** *Given the inner product $\langle f, g\rangle = f(1)g(1) + f(0)g(0)$ on the vector space P_2, answer the following questions.*

   (a) **(5 points)** *Determine the projection of x^2 onto $2x - 1$.*

$$\mathrm{proj}_{2x-1}(x^2) = \frac{\langle 2x-1, x^2\rangle}{\langle 2x-1, 2x-1\rangle}(2x-1) = \frac{1\cdot 1 + (-1)\cdot 0}{1\cdot 1 + (-1)\cdot(-1)}(2x-1) = \frac{1}{2}(2x-1) = x - \frac{1}{2}$$

   (b) **(10 points)** *Find a nonzero vector which is orthogonal to $x + 3$.*
   We want a function $f(x)$ such that $\langle f(x), x+3\rangle = 0$. Using the given inner rpoduct, this requires that $f(1)\cdot 4 + f(0)\cdot 3 = 0$. In other words, $f(1) = \frac{-4}{3}f(0)$. There are many quadratic or lower-degree functions which achieve this goal; for instance, we might try to build a linear function such that $f(0) = 4$ and $f(1) = -3$. This would be $f(x) = 4 - 7x$.

5. **(10 points)** *Answer the following questions.*

   (a) **(5 points)** *What is the distance from the point* $(1, 6)$ *to the line* $y = -2x + 5$*?*
   Translating downwards, we might ask instead for the distance from the point $(1, 1)$ to the line $y = -2x$. This distance would be the length of the component of the point-representing vector $\mathbf{u} = (1, 1)^{\mathrm{T}}$ which is orthogonal to the line-representing vector $\mathbf{v} = (1, -2)^{\mathrm{T}}$. We start by taking a projection of the point onto the line:

$$\mathbf{p} = \operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{v}^{\mathrm{T}}\mathbf{u}}{\mathbf{v}^{\mathrm{T}}\mathbf{v}}\mathbf{v} = \frac{-1}{5}(1, -2)^{\mathrm{T}} = \left(\frac{-1}{5}, \frac{2}{5}\right)^{\mathrm{T}}$$

   and then subtract this from the point itself to get the orthogonal component:

$$\mathbf{z} = \mathbf{u} - \mathbf{p} = \left(\frac{6}{5}, \frac{3}{5}\right)$$

   and this component has length $\sqrt{\left(\frac{6}{5}\right)^2 + \left(\frac{3}{5}\right)^2} = \frac{3}{5}\sqrt{5}$.

   (b) **(5 points)** *What is the distance from* $(0, 1, 1)$ *to the plane* $4x - 5y + z = 0$*?*
   Here we want to project the point represented by the vector $\mathbf{u} = (0, 1, 1)^{\mathrm{T}}$ onto the normal line to the plane, described by the vector $\mathbf{v} = (4, -5, 1)^{\mathrm{T}}$.

$$\mathbf{p} = \operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{v}^{\mathrm{T}}\mathbf{u}}{\mathbf{v}^{\mathrm{T}}\mathbf{v}}\mathbf{v} = \frac{-4}{42}(4, -5, 1)^{\mathrm{T}}$$

   , and calculate its length to get

$$\|\mathbf{p}\| = \left|\frac{-4|}{42}\right| \sqrt{4^2 + (-5)^2 + 1^2} = \frac{4\sqrt{42}}{42}.$$

6. **(15 points)** *Answer the following questions about the linear transformation* L *on* $\mathbf{R}^3$ *here described:* $L((x, y, z)^{\mathrm{T}}) = (2x - y, y + z, x + 2z)^{\mathrm{T}}$.

   (a) **(5 points)** *What matrix represents* L *with respect to the standard basis* $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$*?*
   By noting the images of each of the basis vectors, and using them as columns, we get the matrix representation $A = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$.

   (b) **(5 points)** *For* $\mathbf{u}_1 = (0, 1, 1)^{\mathrm{T}}$*,* $\mathbf{u}_2 = (2, 1, 0)^{\mathrm{T}}$*, and* $\mathbf{u}_3 = (1, 1, 0)^{\mathrm{T}}$*, what matrix represents* L *with respect to the nonstandard basis* $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$*?*
   We might determine the image of each of the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, or simply apply a change-of-basis matrix to the matrix A discovered above. Using the first approach:

$$L(\mathbf{u}_1) = (-1, 2, 2)^{\mathrm{T}} = 2\mathbf{u}_1 - \mathbf{u}_2 + \mathbf{u}_3$$

$$L(\mathbf{u}_2) = (3, 1, 2)^{\mathrm{T}} = 2\mathbf{u}_1 + 4\mathbf{u}_2 - 5\mathbf{u}_3$$

$$L(\mathbf{u}_3) = (1, 1, 1)^{\mathrm{T}} = \mathbf{u}_1 + \mathbf{u}_2 - \mathbf{u}_3$$

   which gives the matrix $B = \begin{pmatrix} 2 & 2 & 1 \\ -1 & 4 & 1 \\ 1 & -5 & -1 \end{pmatrix}$.

Using a change-of-basis matrix, we have

$$B = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 \\ -1 & 4 & 1 \\ 1 & -5 & -1 \end{pmatrix}.$$

(c) **(5 points)** *Determine a basis for the kernel of L.*
The kernel of L is just the nullspace of A. Either by row-reducing A or by calculating its determinant, we may see that it is nonsingular, so the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$, which is to say that the nullspace of A is zero-dimensional, with an empty basis.

7. **(15 points)** *Use the Gram-Schmidt process to orthonormalize the basis* $\{(3,4,0)^{\mathrm{T}}, (1,2,-1)^{\mathrm{T}}, (0,1,1)^{\mathrm{T}}\}$.

We normalize the first vector (which we may call $\mathbf{u}_1$ to get our first normalized vector:

$$\mathbf{q}_1 = \frac{1}{\|\mathbf{u}_1\|}\mathbf{u}_1 = \begin{pmatrix} \frac{3}{4} \\ \frac{4}{5} \\ 0 \end{pmatrix}.$$

Next we determine the projection of $\mathbf{u}_2$ onto $\mathbf{q}_1$:

$$\mathbf{p}_2 = (\mathbf{q}_1^{\mathrm{T}}\mathbf{u}_2)\mathbf{q}_1 = (\tfrac{33}{25}, \tfrac{44}{25}, 0)^{\mathrm{T}}$$

and then subtract it from $\mathbf{u}_2$ to get the orthogonal component:

$$\mathbf{u}_2 - \mathbf{p}_2 = (\tfrac{-8}{25}, \tfrac{6}{25}, -1)^{\mathrm{T}}$$

which we then normalize:

$$\mathbf{q}_2 = \frac{1}{\|\mathbf{u}_2 - \mathbf{p}_2\|}(\mathbf{u}_2 - \mathbf{p}_2) = \begin{pmatrix} \frac{-8}{\sqrt{725}} \\ \frac{6}{\sqrt{725}} \\ \frac{-25}{\sqrt{725}} \end{pmatrix}$$

and finally, we project $\mathbf{u}_3$ onto $\mathbf{q}_1$ and $\mathbf{q}_2$:

$$\mathbf{p}_3 = (\mathbf{q}_1^{\mathrm{T}}\mathbf{u}_3)\mathbf{q}_1 + (\mathbf{q}_2^{\mathrm{T}}\mathbf{u}_3)\mathbf{q}_2 = (\tfrac{12}{25}, \tfrac{16}{25}, 0)^{\mathrm{T}} + (\tfrac{-8(-19)}{725}, \tfrac{6(-19)}{725}, \tfrac{25(-19)}{725})^{\mathrm{T}} = (\tfrac{20}{29}, \tfrac{14}{29}, \tfrac{19}{29})^{\mathrm{T}}$$

$$\mathbf{u}_3 - \mathbf{p}_3 = (\tfrac{-20}{29}, \tfrac{15}{29}, \tfrac{-10}{29})^{\mathrm{T}}$$

$$\mathbf{q}_3 = \frac{1}{\|\mathbf{u}_3 - \mathbf{p}_3\|}(\mathbf{u}_3 - \mathbf{p}_3) = \begin{pmatrix} \frac{-4}{\sqrt{29}} \\ \frac{3}{\sqrt{29}} \\ \frac{-2}{\sqrt{29}} \end{pmatrix}.$$